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EE263 (Introduction to Linear Dynamical Systems)
Problem Session 6
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1 Optimal choice of initial temperature profile

Consider a thermal system described by an n-element finite-element model. The elements are arranged in a line, with the temperature of element i at time t denoted $T_i(t)$. Temperature in °C above the ambient; in particular, a negative value of $T_i(t)$ indicates a temperature that is below the ambient temperature. The dynamics of the system are described by the following system of differential equations:

$$\begin{aligned} c_1\dot{T}_1 &= -a_1T_1 - b_1(T_1 - T_2), \\ c_i\dot{T}_i &= -a_iT_i - b_i(T_i - T_{i+1}) - b_{i-1}(T_i - T_{i-1}), \qquad i = 2, \ldots, n-1, \\ c_n\dot{T}_n &= -a_nT_n - b_{n-1}(T_n - T_{n-1}), \end{aligned}$$

where $c \in \mathbb{R}^n$, $a \in \mathbb{R}^n$, and $b \in \mathbb{R}^n$ are given positive constants.

We can interpret this model as follows. The parameter c_i is the heat capacity of element i, so $c_i\dot{T}_i$ is the net heat flow into element i. The parameter a_i gives the thermal conductance between element i, and the environment, so a_iT_i is the heat flow from element i to the environment (that is, the direct heat loss from element i). The parameter b_i gives the thermal conductance between elements i and $i+1$, so $b_i(T_i - T_{i+1})$ is the heat flow from element i to element $i+1$. Similarly, $b_{i-1}(T_i - T_{i-1}$ is the heat flow from element i to element $i-1$.

Your objective is to choose the initial temperature profile, $T(0) \in \mathbb{R}^n$, so that $T(t^{\text{des}}) \approx T^{\text{des}}$, where $t^{\text{des}} > 0$ is a given time when we want the temperature profile to be as close as possible to a given temperature profile $T^{\text{des}} \in \mathbb{R}^n$. Additionally, $\|T(0)\|$ cannot be too large (for example, it may take a lot of energy to establish such a profile, or such an extreme profile may damage the thermal system). More concretely, you want to minimize

$$\frac{\|T(t^{\text{des}}) - T^{\text{des}}\|}{\sqrt{n}},$$

which is the root mean squared deviation between the temperature profile at time t^{des}, and the desired temperature profile. Additionally, you have the constraint

$$\frac{\|T(0)\|}{\sqrt{n}} \leq T^{\max},$$

which says that the root mean squared deviation of the initial temperature profile from the ambient temperature can be at most $T^{\max}$.

(a) Explain how to find the value of $T(0)$ that minimizes the objective subject to the constraint.

(b) Apply your method to the data given in `temperature_profile_data.m`. Report the RMS temperature error,

$$\frac{\|T(t^{\text{des}}) - T^{\text{des}}\|}{\sqrt{n}},$$

and the RMS deviation of the initial temperature profile from the ambient,

$$\frac{\|T(0)\|}{\sqrt{n}}.$$

Plot $T(0)$, $T(t^{\text{des}})$, and T^{des} on a single set of axes.

2 Output-response envelope with uncertain initial condition

Consider the autonomous, continuous-time linear dynamical system

$$\dot{x} = Ax, \qquad y(t) = Cx(t),$$

where $x(t) \in \mathbb{R}^n$, and $y(t) \in \mathbb{R}$. We do not know the initial condition $x(0)$ exactly, but we do know that it lies in the ball of radius r centered at x_0: that is,

$$\|x(0) - x_0\| \leq r.$$

We call x_0 the nominal initial condition, and the resulting output $y_0(t) = Ce^{tA}x_0$ the nominal output. We define the maximum output, or upper output envelope, as

$$\bar{y}(t) = \max\{y(t) : \|x(0) - x_0\| \leq r\}.$$

In other words, $\bar{y}(t)$ is the maximum possible value of the output at time t over all possible initial conditions. (Note that the maximum possible value of the output at time t may be achieved with a different initial condition for each t.) Similarly, we define the minimum output, or lower output envelope, as

$$\underline{y}(t) = \min\{y(t) : \|x(0) - x_0\| \leq r\}.$$

In other words, $\bar{y}(t)$ is the minimum possible value of the output at time t over all possible initial conditions.

(a) Explain how to find $\bar{y}(t)$, and $\underline{y}(t)$ given A, C, x_0, and r.

(b) Apply your method to the data given in `output_envelope_data.m`. Plot $y_0(t)$, $\bar{y}(t)$ and $\underline{y}(t)$ on a single set of axes for $0 \leq t \leq 10$.

3 Closed walks in a directed graph

Consider a directed graph with nodes $1, \ldots, n$, and adjacency matrix $A \in \mathbb{R}^{n \times n}$, where

$$A_{ij} = \begin{cases} 1 & \text{there is an edge from node } j \text{ to node } i, \\ 0 & \text{otherwise.} \end{cases}$$

A closed walk of length L is a sequence of nodes $n_1, \ldots, n_{L+1}$ such that there is a directed edge from n_i to n_{i+1} for $i = 1, \ldots, L$, and $n_1 = n_{L+1}$. Let $N_L(i)$ denote the number of closed walks of length L that start and end at node i. Then, the total number of closed walks of length L is $\sum_{i=1}^{n} N_L(i)$.

Suppose that A is diagonalizable, has a unique dominant eigenvalue λ_1, and that λ_1 is real and positive. (Recall that an eigenvalue λ of A is said to be dominant if $|\lambda| \geq |\tilde{\lambda}|$ for every eigenvalue $\tilde{\lambda}$ of A.)

(a) For a given value of L, explain how to find a node i that maximizes $N_L(i)$.

(b) Define $G(i)$ to be

$$G(i) = \lim_{L\to\infty} \frac{N_L(i)}{\sum_{j=1}^{n} N_L(j)}.$$

Explain how to find a node i that maximizes $G(i)$.

(c) The file `closed_walks_data.m` defines an adjacency matrix A. For this example, find a node $i^{(5)}$ that maximizes $N_5(i)$, and a node $i^{(\infty)}$ that maximizes $G(i)$. (You only need to find one such node in each case, although there may be multiple nodes that achieve the maximum.) Report $i^{(5)}$, $N_5(i^{(5)})$, $i^{(\infty)}$, and $G(i^{(\infty)})$.

4 Minimum-energy control

Consider the discrete-time linear dynamical system

$$x(t+1) = Ax(t) + Bu(t), \qquad t = 0, 1, 2, \ldots,$$

where $x(t) \in \mathbb{R}^n$, and $u(t) \in \mathbb{R}$. The initial state is $x(0) = 0$.

(a) Find the matrix $\mathcal{C}_T$ such that

$$x(T) = \mathcal{C}_T \begin{bmatrix} u(T-1) \\ \vdots \\ u(1) \\ u(0) \end{bmatrix}.$$

(b) For the rest of the problem, consider the specific system with

$$A = \begin{bmatrix} 0.5 & 0.7 & -0.9 & -0.5 \\ 0.4 & -0.7 & 0.1 & 0.3 \\ 0.7 & 0.0 & -0.6 & 0.1 \\ 0.4 & -0.1 & 0.8 & -0.5 \end{bmatrix}, \qquad \text{and} \qquad B = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Suppose we want the state at time T to be

$$x_{\text{des}}^{(1)} = \begin{bmatrix} 0.8 \\ 2.3 \\ -0.7 \\ -0.3 \end{bmatrix}.$$

What is the smallest value of T for which there is a corresponding input sequence $u(0), \ldots, u(T-1)$ such that $x(T) = x_{\text{des}}^{(1)}$? For this smallest value of T, what is an input sequence $u(0), \ldots, u(T-1)$ such that $x(T) = x_{\text{des}}^{(1)}$?

What is the smallest value of T such that, for every $x_{\text{des}} \in \mathbb{R}^n$, we can find an input sequence $u(0), \ldots, u(T-1)$ such that $x(T) = x_{\text{des}}$? We call this value $T_{\min}$.

(c) The total energy of the input sequence $u(0), \dots, u(T-1)$ is

$$E(T) = \sum_{t=0}^{T-1} u(t)^2.$$

For a given time $T \geq T_{\min}$, and desired state $x_{\text{des}} \in \mathbb{R}^n$, explain how to find the minimum-energy input sequence that achieves $x(T) = x_{\text{des}}$. Now consider the specific desired state

$$x_{\text{des}}^{(2)} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

For $T = T_{\min}, \dots, 30$, find the minimum-energy input sequence that achieves $x(T) = x_{\text{des}}^{(2)}$. For each T, let $E_{\min}(T)$ denote the corresponding minimum input energy. Plot $E_{\min}(T)$ versus T.

(d) You should observe that $E_{\min}(T)$ is a nonincreasing function of T. Show that this is true for arbitrary values of A, B and x_{des}.