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EE263 (Introduction to Linear Dynamical Systems)
Problem Session 7
July 31, 2013

1 Finding worst-case inputs

Consider the single-input, single-output system

$$x(t+1) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t),$$

where $x(0) = 0$, and

$$A = \begin{bmatrix} 0.9 & 0.5 \\ -0.5 & 0.7 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \text{and} \qquad C = \begin{bmatrix} 1 & 2 \end{bmatrix}.$$

This system is a very simple model of a building. The input u is the ground displacement (for example, during an earthquake), and y is the displacement of the top of the building. We know that the input u satisfies

$$\sum_{t=0}^{49} u(t)^2 \leq 1 \qquad \text{and} \qquad u(t) = 0, \quad t \geq 50.$$

We interpret these conditions as the earthquake being known to have at most unit energy, and duration at most 50 samples.

(a) How large can $\sum_{t=0}^{99} y(t)^2$ be? Plot an input u that maximizes $\sum_{t=0}^{99} y(t)^2$, along with the resulting output y.

(b) How large can $|y(100)|$ be? Plot an input u that maximizes $|y(100)|$, along with the resulting output y.

2 2D projection with minimum distance distortion

We want to visualize a set of data points $a_1, \ldots, a_N \in \mathbb{R}^n$, where $n > 2$. To do this, we form the coordinates $c_i = Q^\mathsf{T} a_i \in \mathbb{R}^2$ for $i = 1, \ldots, N$, where $Q \in \mathbb{R}^{n \times 2}$ has orthonormal columns; then, we plot the coordinates on a screen. This may allow us to see or recognize some structure in the data points that would be hard to recognize directly from the original data. This problem concerns the choice of the matrix Q.

Let $D_{ij} = \|a_i - a_j\|$ be the distance between the data points a_i and a_j in the original data set, and let $\tilde{D}_{ij} = \|c_i - c_j\|$ be the distance between the coordinates c_i and c_j. Ideally, we want $D_{ij} = \tilde{D}_{ij}$ for all i and j. However, this is not possible in most cases, so we must settle for an approximation. Since the columns of Q are orthonormal, we have that $\tilde{D}_{ij} \leq D_{ij}$ for all i and j. Thus, it seems reasonable to choose Q in order to maximize

$$J = \sum_{i=1}^{N} \sum_{j=1}^{N} \tilde{D}_{ij}^2.$$

Intuitively, this drives $\tilde{D}_{ij}$ towards D_{ij}, which is our goal. Note that the solution is never unique: if Q is one solution, then QZ is another solution, where $Z \in \mathbb{R}^{2\times 2}$ is any orthogonal matrix. (Using QZ instead of Q applies a rotation or reflection to the coordinates; this does not affect the pairwise distances.)

(a) Explain how to find Q. You may assume the data points span $\mathbb{R}^n$.

(b) Carry out your method on the data given in `projection_2d_data.m`. Plot the coordinates with an optimal choice of Q.

3 Detecting a linear relationship

Suppose we have N measurements $y_1, \ldots, y_N \in \mathbb{R}^n$ of a signal $x_1, \ldots, x_N \in \mathbb{R}^n$:

$$y_i = x_i + d_i, \qquad i = 1, \ldots, N,$$

where d_i is measurement noise, and is presumed to be small. We hypothesize that there is a linear relationship among the components of x: that is, there is a nonzero vector $q \in \mathbb{R}^n$ such that $q^\mathsf{T} x_i = 0$ for $i = 1, \ldots, N$. The geometric interpretation of such a relationship is that all of the vectors x_i line in the hyperplane $q^\mathsf{T} x = 0$. In order to remove some of the ambiguity in q, we will assume that $\|q\| = 0$, Even if there is a linear relationship among the components of x, there will usually not be a linear relationship among the components of y: $q^\mathsf{T} y_i = q^\mathsf{T} d_i$, which is small if d_i is small, but will not typically be exactly equal to zero. Geometrically, we want all of the y_i to be close to the hyperplane with normal vector q. Define the normalized error measure

$$\rho = \frac{\sqrt{\frac{1}{N}\sum_{i=1}^N (q^\mathsf{T} y_i)^2}}{\sqrt{\frac{1}{N}\sum_{i=1}^N \|y_i\|^2}}.$$

We can think of ρ as the root mean squared distance of the y_i to the hyperplane with normal vector q divided by the root mean squared length of the vectors y_i.

(a) Explain how to find the unit vector q that minimizes ρ.

(b) Apply your method to the data given in `detect_linear_relationship_data.m`. Report the vector q that you find, and the corresponding value of ρ.