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EE263 (Introduction to Linear Dynamical Systems)
Problem Session 8
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1 Linear system with a quadrant detector

In this problem we consider the specific system

$$\dot{x} = Ax,$$

where

$$A = \begin{bmatrix} 0.5 & 1.4 \\ -0.7 & 0.5 \end{bmatrix}.$$

The initial state $x(0)$ is unknown. We have a sensor that gives us the sign of each component of the state at discrete times $t = 0, 1, 2, \ldots$:

$$y_1(t) = \operatorname{sgn}(x_1(t)), \qquad y_2(t) = \operatorname{sgn}(x_2(t)).$$

You can think of $y(t) = (y_1(t), y_2(t))$ as a measurement of the quadrant the state is in at time t (hence the name quadrant detector). You observe the sensor measurements

$$y(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad \text{and} \qquad y(1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Based on these measurements, what are the possible values of $y(2)$? In other words, if $x(0)$ and $x(1)$ are both in the fourth quadrant, what quadrant can $x(2)$ be in?

2 Blind signal detection

Consider a binary signal $s_1, \ldots, s_T \in \{-1, +1\}$. This signal is sent through a communication channel to a receiver, which measures the signal

$$y_t = a s_t + v_t, \qquad t = 1, \ldots, T,$$

where $a \in \mathbb{R}^n$ is the channel response, and $v_1, \ldots, v_T \in \mathbb{R}^n$ is noise. We will assume that a is nonzero but otherwise unknown, and the noise is centered around zero.

The receiver constructs an estimate of the transmitted signal:

$$\hat{s}_t = w^\mathsf{T} y_t, \qquad t = 1, \ldots, T,$$

where $w \in \mathbb{R}^n$ is a weight vector. You want to choose w so that $\hat{s}_t \approx s_t$ for $t = 1, \ldots, T$. If you knew a, then a reasonable choice for w would be $w = A^\dagger = \frac{1}{\|a\|^2} a$; this is the smallest vector w such that $w^\mathsf{T} a = 1$.

However, we do not know a. Using the received signal to estimate the transmitted signal when the mapping from the transmitted signal to the received signal is unknown is called blind signal estimation or blind signal detection. Here is one approach to the problem. Ignoring the noise signal, if we choose w so that $w^\mathsf{T} y_t \approx s_t$, then we have that

$$\frac{1}{T} \sum_{t=1}^{T} (w^\mathsf{T} y_t)^2 \approx 1.$$

Since $w^\mathsf{T} v_t$ gives the noise contribution to $\hat{s}_t$, we want w to be as small as possible. Thus, we choose w in order to minimize $\|w\|$ subject to the constraint that

$$\frac{1}{T}\sum_{t=1}^{T}(w^\mathsf{T} y_t)^2 = 1.$$

This problem does not have a unique solution: if w is one solution, then $-w$ is another solution. Therefore, we can only hope to recover either an approximation of s_t or an approximation of $-s_t$.

(a) Explain how to find w given the received signal $y_1, \ldots, y_T$.

(b) Apply your method to the signal in `blind_detection_data.m`. Report your weight vector. Plot a histogram of the values of $w^\mathsf{T} y_t$. Once you have chosen w, a reasonable estimate of s_t (or possibly $-s_t$) is

$$\tilde{s}_t = (w^\mathsf{T} y_t), \qquad t = 1, \ldots, T.$$

The file `blind_detection_data.m` also contains the transmitted signal, $s_1, \ldots, s_T$. Give your error rate: that is, the fraction of times t for which $\tilde{s}_t \neq s_t$. (If your error rate is more than 50 %, you should negate w.)

3 Determining the number of signal sources

The signals transmitted by n sources are measured at m receivers at times $t = 1, \ldots, p$. Let $x_j(t) \in \mathbb{R}$ denote the signal transmitted by the jth receiver at time t, and let $y_i(t) \in \mathbb{R}$ denote the signal measured at the ith receiver at time t. let $a_{ij} \in \mathbb{R}$ denote the gain from transmitter j to receiver i. We can compactly describe this situation by the equation

$$y(t) = Ax(t) + v(t), \qquad t = 1, \ldots, p,$$

where $v(t) \in \mathbb{R}^m$ represents measurement noise. We do not know the gains a_{ij}, the transmitted signal $x(t)$, nor even the number of sources n. However, we do make the following assumptions.

- There are fewer sources than receivers: that is, $n < m$.
- The matrix A is full-rank, and well-conditioned.
- All of the sources have approximately the same average power, the signal $x(t)$ is unpredictable, and the source signals are all unrelated to one another.
- The sensor noise $v(t)$ is small relative to the received signal $Ax(t)$.

(a) Given measurements $y(1), \ldots, y(p) \in \mathbb{R}^m$, explain how to estimate the number of sources n.

(b) The file `number_signal_sources.m` defines the following variables.

  - `m`, the number of receivers

- `p`, the number of measurements
- `Y`, an $m \times p$ matrix whose tth column is $y(t)$

Apply your method to this data, and report your guess of the number of sources.

This problem is intentionally vague, and there may not be a single "correct" answer. We want to see if you can take an open-ended problem, and come up with a reasonable approach.