Lecture 11
Eigenvectors and diagonalization

- eigenvectors
- dynamic interpretation: invariant sets
- complex eigenvectors & invariant planes
- left eigenvectors
- diagonalization
- modal form
- discrete-time stability

Eigenvectors and eigenvalues

$\lambda \in \mathbf{C}$ is an *eigenvalue* of $A \in \mathbf{C}^{n\times n}$ if

$$\mathcal{X}(\lambda) = \det(\lambda I - A) = 0$$

equivalent to:

- there exists nonzero $v \in \mathbf{C}^n$ s.t. $(\lambda I - A)v = 0$, $i.e.$,

  $$Av = \lambda v$$

  any such v is called an *eigenvector* of A (associated with eigenvalue λ)

- there exists nonzero $w \in \mathbf{C}^n$ s.t. $w^T(\lambda I - A) = 0$, $i.e.$,

  $$w^T A = \lambda w^T$$

  any such w is called a *left eigenvector* of A

- if v is an eigenvector of A with eigenvalue λ, then so is αv, for any $\alpha \in \mathbf{C}$, $\alpha \neq 0$

- even when A is real, eigenvalue λ and eigenvector v can be complex

- when A and λ are real, we can always find a real eigenvector v associated with λ: if $Av = \lambda v$, with $A \in \mathbf{R}^{n \times n}$, $\lambda \in \mathbf{R}$, and $v \in \mathbf{C}^n$, then

$$A \Re v = \lambda \Re v, \qquad A \Im v = \lambda \Im v$$

  so $\Re v$ and $\Im v$ are real eigenvectors, if they are nonzero
  (and at least one is)

- *conjugate symmetry*: if A is real and $v \in \mathbf{C}^n$ is an eigenvector associated with $\lambda \in \mathbf{C}$, then $\overline{v}$ is an eigenvector associated with $\overline{\lambda}$:

  taking conjugate of $Av = \lambda v$ we get $\overline{Av} = \overline{\lambda v}$, so

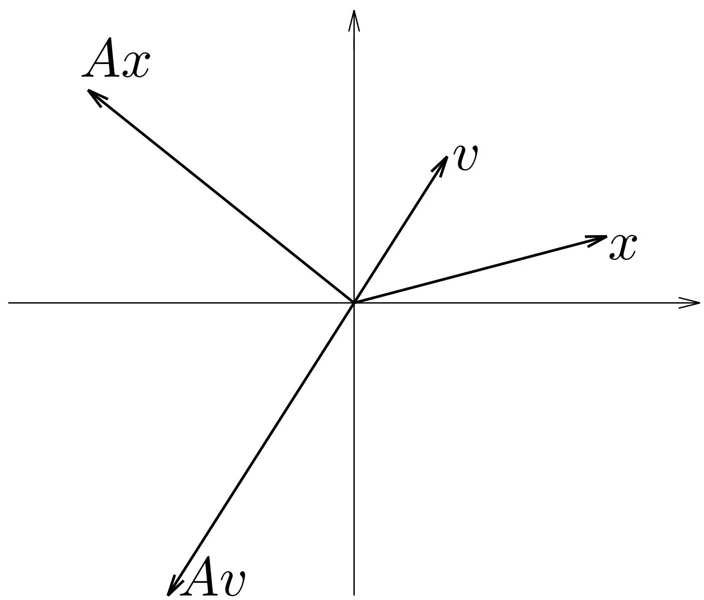
$$A \overline{v} = \overline{\lambda} \overline{v}$$

we'll assume A is real from now on . . .

Scaling interpretation

(assume $\lambda \in \mathbf{R}$ for now; we'll consider $\lambda \in \mathbf{C}$ later)

if v is an eigenvector, effect of A on v is very simple: scaling by λ



(what is λ here?)

- $\lambda \in \mathbf{R}$, $\lambda > 0$: v and Av point in same direction

- $\lambda \in \mathbf{R}$, $\lambda < 0$: v and Av point in opposite directions

- $\lambda \in \mathbf{R}$, $|\lambda| < 1$: Av smaller than v

- $\lambda \in \mathbf{R}$, $|\lambda| > 1$: Av larger than v

(we'll see later how this relates to stability of continuous- and discrete-time systems. . .)

Dynamic interpretation

suppose $Av = \lambda v$, $v \neq 0$

if $\dot{x} = Ax$ and $x(0) = v$, then $x(t) = e^{\lambda t} v$

several ways to see this, $e.g.$,

$$\begin{aligned} x(t) = e^{tA} v &= \left(I + tA + \frac{(tA)^2}{2!} + \cdots \right) v \\ &= v + \lambda t v + \frac{(\lambda t)^2}{2!} v + \cdots \\ &= e^{\lambda t} v \end{aligned}$$

(since $(tA)^k v = (\lambda t)^k v$)

- for $\lambda \in \mathbf{C}$, solution is complex (we'll interpret later); for now, assume $\lambda \in \mathbf{R}$

- if initial state is an eigenvector v, resulting motion is very simple — always on the line spanned by v

- solution $x(t) = e^{\lambda t} v$ is called *mode* of system $\dot{x} = Ax$ (associated with eigenvalue λ)

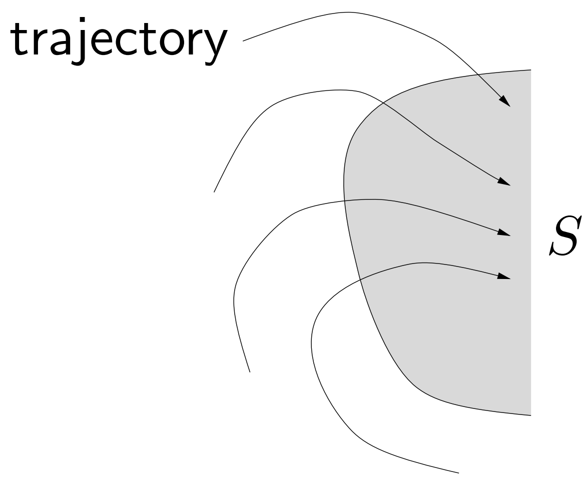
- for $\lambda \in \mathbf{R}$, $\lambda < 0$, mode contracts or shrinks as $t \uparrow$

- for $\lambda \in \mathbf{R}$, $\lambda > 0$, mode expands or grows as $t \uparrow$

Invariant sets

a set $S \subseteq \mathbf{R}^n$ is *invariant* under $\dot{x} = Ax$ if whenever $x(t) \in S$, then $x(\tau) \in S$ for all $\tau \geq t$

$i.e.$: once trajectory enters S, it stays in S



vector field interpretation: trajectories only cut *into* S, never out suppose $Av = \lambda v$, $v \neq 0$, $\lambda \in \mathbf{R}$

- line $\{ \, tv \mid t \in \mathbf{R} \, \}$ is invariant

  (in fact, ray $\{ \, tv \mid t > 0 \, \}$ is invariant)

- if $\lambda < 0$, line segment $\{ \, tv \mid 0 \leq t \leq a \, \}$ is invariant

Complex eigenvectors

suppose $Av = \lambda v$, $v \neq 0$, λ is complex

for $a \in \mathbf{C}$, (complex) trajectory $ae^{\lambda t}v$ satisfies $\dot{x} = Ax$

hence so does (real) trajectory

$$\begin{aligned} x(t) &= \Re\left(ae^{\lambda t}v\right) \\ &= e^{\sigma t}\left[\begin{array}{cc} v_{\mathrm{re}} & v_{\mathrm{im}} \end{array}\right]\left[\begin{array}{cc} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{array}\right]\left[\begin{array}{c} \alpha \\ -\beta \end{array}\right] \end{aligned}$$

where

$$v = v_{\mathrm{re}} + iv_{\mathrm{im}}, \quad \lambda = \sigma + i\omega, \quad a = \alpha + i\beta$$

- trajectory stays in *invariant plane* $\operatorname{span}\{v_{\mathrm{re}}, v_{\mathrm{im}}\}$
- σ gives logarithmic growth/decay factor
- ω gives angular velocity of rotation in plane

Dynamic interpretation: left eigenvectors

suppose $w^T A = \lambda w^T$, $w \neq 0$

then
$$\frac{d}{dt}(w^T x) = w^T \dot{x} = w^T A x = \lambda (w^T x)$$
$i.e.$, $w^T x$ satisfies the DE $d(w^T x)/dt = \lambda (w^T x)$

hence $w^T x(t) = e^{\lambda t} w^T x(0)$

- even if trajectory x is complicated, $w^T x$ is simple
- if, $e.g.$, $\lambda \in \mathbf{R}$, $\lambda < 0$, halfspace $\{ z \mid w^T z \leq a \}$ is invariant (for $a \geq 0$)
- for $\lambda = \sigma + i\omega \in \mathbf{C}$, $(\Re w)^T x$ and $(\Im w)^T x$ both have form

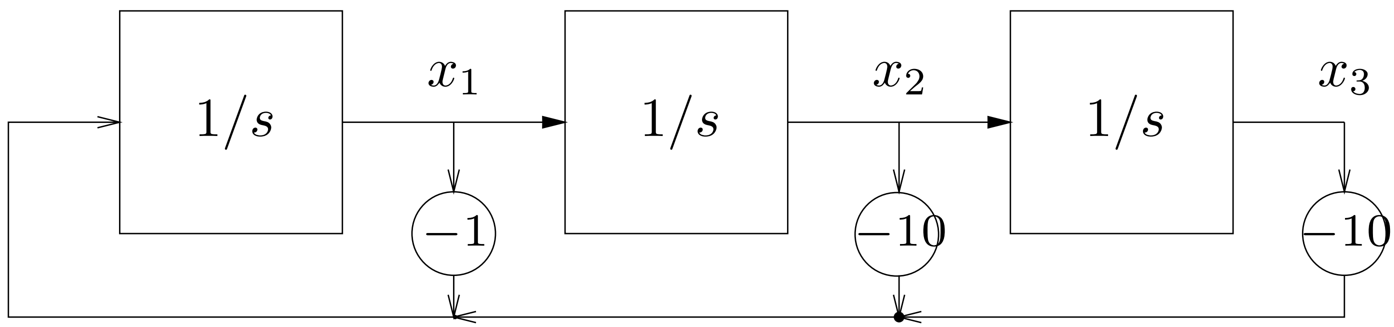
$$e^{\sigma t} \left(\alpha \cos(\omega t) + \beta \sin(\omega t) \right)$$

Summary

- *right eigenvectors* are initial conditions from which resulting motion is simple ($i.e.$, remains on line or in plane)
- *left eigenvectors* give linear functions of state that are simple, for any initial condition

example 1: $\dot{x} = \begin{bmatrix} -1 & -10 & -10 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x$

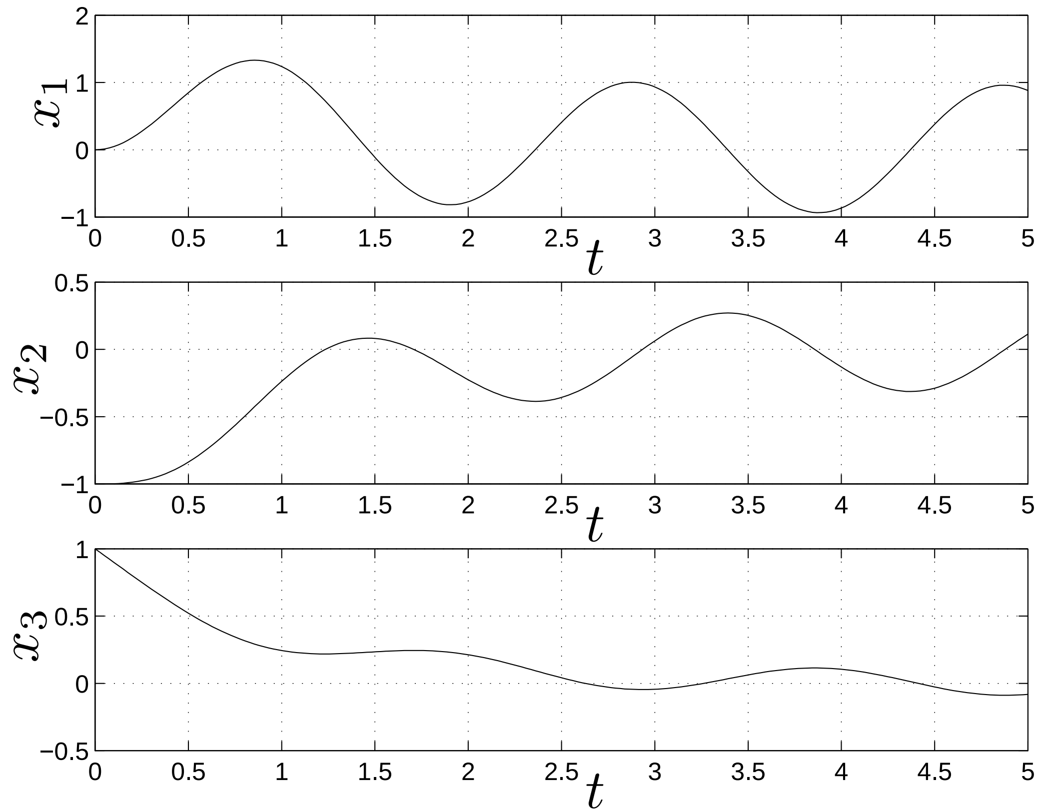
block diagram:



$$\mathcal{X}(s) = s^3 + s^2 + 10s + 10 = (s+1)(s^2+10)$$

eigenvalues are $-1, \ \pm i\sqrt{10}$

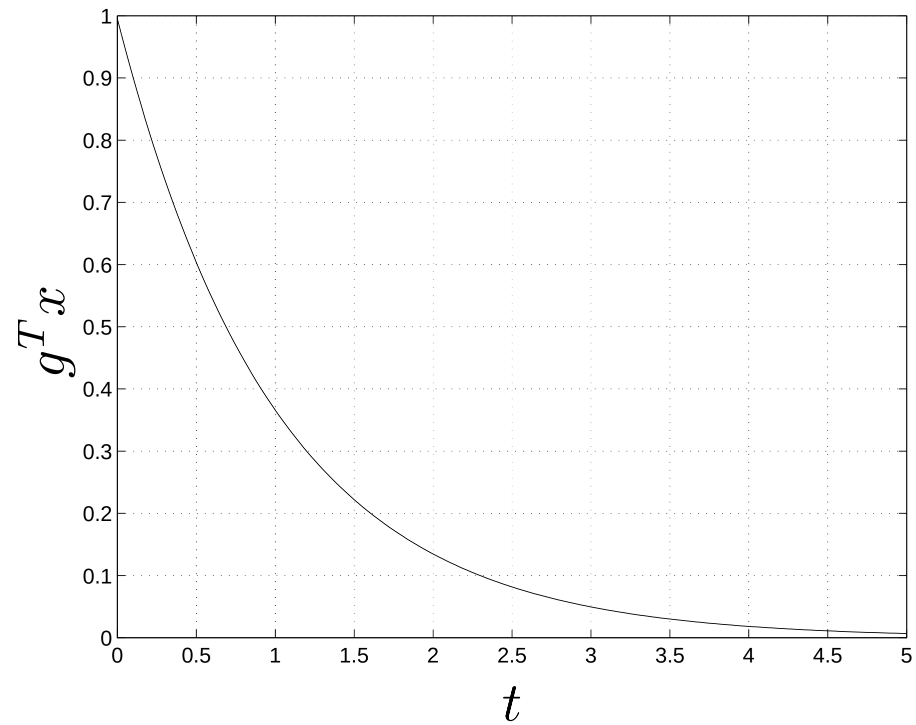
trajectory with $x(0) = (0, -1, 1)$:



left eigenvector associated with eigenvalue -1 is

$$g = \begin{bmatrix} 0.1 \\ 0 \\ 1 \end{bmatrix}$$

let's check $g^T x(t)$ when $x(0) = (0, -1, 1)$ (as above):



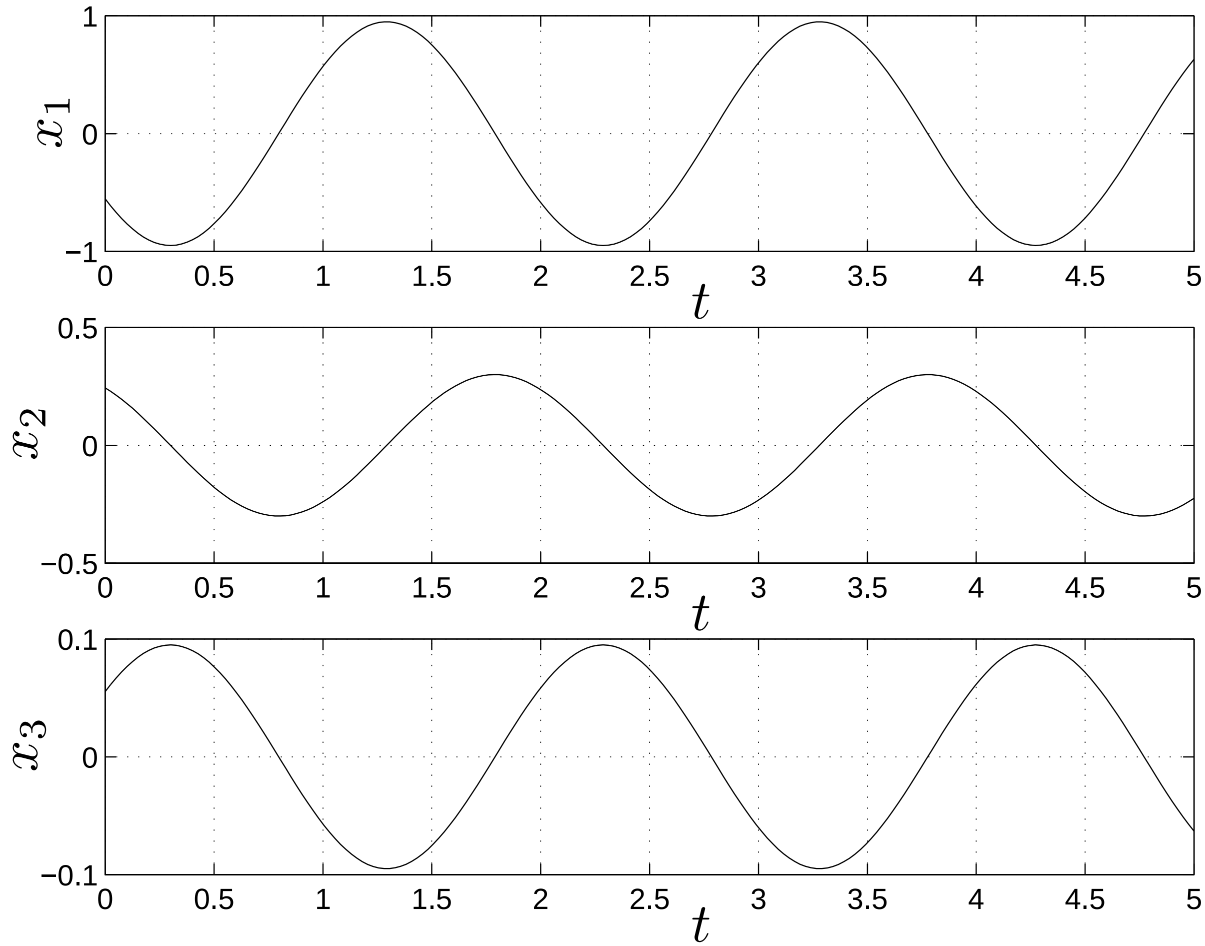
eigenvector associated with eigenvalue $i\sqrt{10}$ is

$$v = \begin{bmatrix} -0.554 + i0.771 \\ 0.244 + i0.175 \\ 0.055 - i0.077 \end{bmatrix}$$

so an invariant plane is spanned by

$$v_{\mathrm{re}} = \begin{bmatrix} -0.554 \\ 0.244 \\ 0.055 \end{bmatrix}, \quad v_{\mathrm{im}} = \begin{bmatrix} 0.771 \\ 0.175 \\ -0.077 \end{bmatrix}$$

for example, with $x(0) = v_{\rm re}$ we have



Example 2: Markov chain

probability distribution satisfies $p(t+1) = Pp(t)$

$p_i(t) = \mathbf{Prob}(\ z(t) = i\ )$ so $\sum_{i=1}^n p_i(t) = 1$

$P_{ij} = \mathbf{Prob}(\ z(t+1) = i \mid z(t) = j\ )$, so $\sum_{i=1}^n P_{ij} = 1$
(such matrices are called *stochastic*)

rewrite as:

$$[1\ 1\ \cdots\ 1]P = [1\ 1\ \cdots\ 1]$$

$i.e.$, $[1\ 1\ \cdots\ 1]$ is a left eigenvector of P with e.v. 1

hence $\det(I - P) = 0$, so there is a right eigenvector $v \neq 0$ with $Pv = v$

it can be shown that v can be chosen so that $v_i \geq 0$, hence we can normalize v so that $\sum_{i=1}^n v_i = 1$

interpretation: v is an *equilibrium distribution*; $i.e.$, if $p(0) = v$ then $p(t) = v$ for all $t \geq 0$

(if v is unique it is called the *steady-state distribution* of the Markov chain)

Diagonalization

suppose $v_1, \ldots, v_n$ is a *linearly independent* set of eigenvectors of $A \in \mathbf{R}^{n \times n}$:

$$Av_i = \lambda_i v_i, \quad i = 1, \ldots, n$$

express as

$$A \left[\begin{array}{ccc} v_1 & \cdots & v_n \end{array} \right] = \left[\begin{array}{ccc} v_1 & \cdots & v_n \end{array} \right] \left[\begin{array}{ccc} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{array} \right]$$

define $T = \left[\begin{array}{ccc} v_1 & \cdots & v_n \end{array} \right]$ and $\Lambda = \mathbf{diag}(\lambda_1, \ldots, \lambda_n)$, so

$$AT = T\Lambda$$

and finally

$$T^{-1}AT = \Lambda$$

- T invertible since $v_1, \ldots, v_n$ linearly independent
- similarity transformation by T *diagonalizes* A

conversely if there is a $T = [v_1 \ \cdots \ v_n]$ s.t.

$$T^{-1}AT = \Lambda = \mathbf{diag}(\lambda_1, \ldots, \lambda_n)$$

then $AT = T\Lambda$, $i.e.$,

$$Av_i = \lambda_i v_i, \quad i = 1, \ldots, n$$

so $v_1, \ldots, v_n$ is a linearly independent set of n eigenvectors of A

we say A is *diagonalizable* if

- there exists T s.t. $T^{-1}AT = \Lambda$ is diagonal
- A has a set of n linearly independent eigenvectors

(if A is not diagonalizable, it is sometimes called *defective*)

Not all matrices are diagonalizable

example: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

characteristic polynomial is $\mathcal{X}(s) = s^2$, so $\lambda = 0$ is only eigenvalue

eigenvectors satisfy $Av = 0v = 0$, *i.e.*

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

so all eigenvectors have form $v = \begin{bmatrix} v_1 \\ 0 \end{bmatrix}$ where $v_1 \neq 0$

thus, A cannot have two independent eigenvectors

Distinct eigenvalues

fact: if A has distinct eigenvalues, *i.e.*, $\lambda_i \neq \lambda_j$ for $i \neq j$, then A is diagonalizable

(the converse is false — A can have repeated eigenvalues but still be diagonalizable)

Diagonalization and left eigenvectors

rewrite $T^{-1}AT = \Lambda$ as $T^{-1}A = \Lambda T^{-1}$, or

$$\begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix} A = \Lambda \begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix}$$

where $w_1^T, \ldots, w_n^T$ are the rows of T^{-1}

thus

$$w_i^T A = \lambda_i w_i^T$$

$i.e.$, the rows of T^{-1} are (lin. indep.) left eigenvectors, normalized so that

$$w_i^T v_j = \delta_{ij}$$

($i.e.$, left & right eigenvectors chosen this way are *dual bases*)

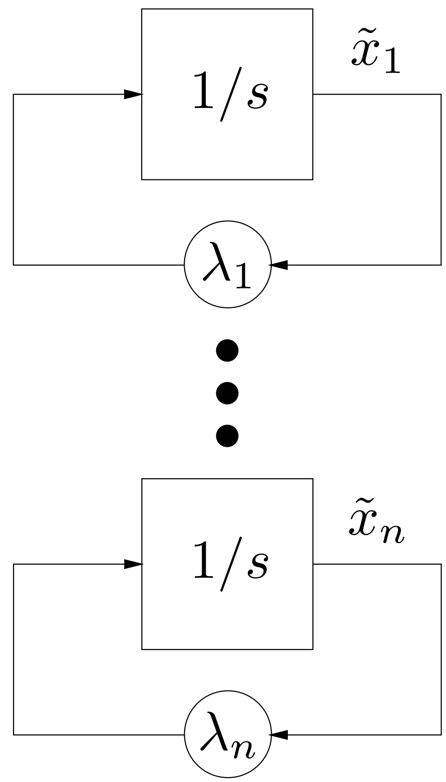
Modal form

suppose A is diagonalizable by T

define new coordinates by $x = T\tilde{x}$, so

$$T\dot{\tilde{x}} = AT\tilde{x} \quad \Leftrightarrow \quad \dot{\tilde{x}} = T^{-1}AT\tilde{x} \quad \Leftrightarrow \quad \dot{\tilde{x}} = \Lambda\tilde{x}$$

in new coordinate system, system is diagonal (decoupled):



trajectories consist of n independent modes, $i.e.$,

$$\tilde{x}_i(t) = e^{\lambda_i t}\tilde{x}_i(0)$$

hence the name *modal form*

Real modal form

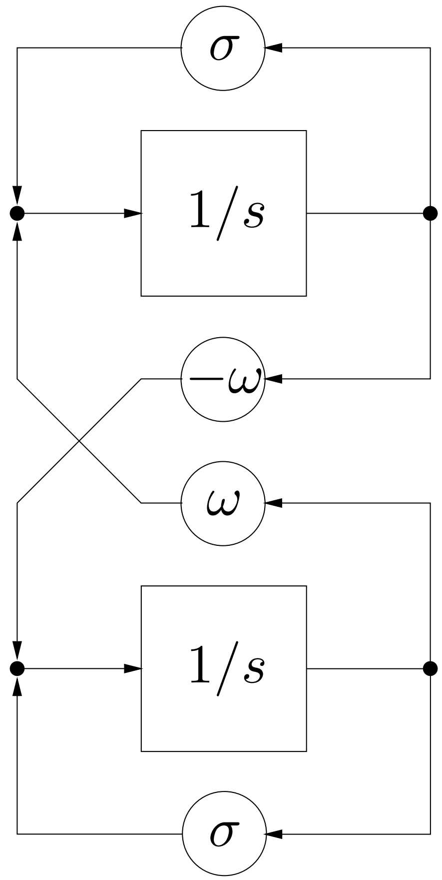
when eigenvalues (hence T) are complex, system can be put in *real modal form*:

$$S^{-1}AS = \mathbf{diag}\left(\Lambda_r, M_{r+1}, M_{r+3}, \ldots, M_{n-1}\right)$$

where $\Lambda_r = \mathbf{diag}(\lambda_1, \ldots, \lambda_r)$ are the real eigenvalues, and

$$M_j = \begin{bmatrix} \sigma_j & \omega_j \\ -\omega_j & \sigma_j \end{bmatrix}, \quad \lambda_j = \sigma_j + i\omega_j, \quad j = r+1, r+3, \ldots, n$$

where λ_j are the complex eigenvalues (one from each conjugate pair) block diagram of 'complex mode':



diagonalization simplifies many matrix expressions

$e.g.$, resolvent:

$$\begin{aligned}(sI-A)^{-1} &= \left(sTT^{-1}-T\Lambda T^{-1}\right)^{-1} \\ &= \left(T(sI-\Lambda)T^{-1}\right)^{-1} \\ &= T(sI-\Lambda)^{-1}T^{-1} \\ &= T\,\mathbf{diag}\left(\frac{1}{s-\lambda_1},\ldots,\frac{1}{s-\lambda_n}\right)T^{-1}\end{aligned}$$

powers ($i.e.$, discrete-time solution):

$$\begin{aligned}A^k &= \left(T\Lambda T^{-1}\right)^k \\ &= \left(T\Lambda T^{-1}\right)\cdots\left(T\Lambda T^{-1}\right) \\ &= T\Lambda^k T^{-1} \\ &= T\,\mathbf{diag}(\lambda_1^k,\ldots,\lambda_n^k)T^{-1}\end{aligned}$$

(for $k<0$ only if A invertible, $i.e.$, all $\lambda_i \neq 0$)

exponential ($i.e.$, continuous-time solution):

$$\begin{aligned} e^A &= I + A + A^2/2! + \cdots \\ &= I + T\Lambda T^{-1} + \left(T\Lambda T^{-1}\right)^2/2! + \cdots \\ &= T(I + \Lambda + \Lambda^2/2! + \cdots)T^{-1} \\ &= Te^{\Lambda}T^{-1} \\ &= T\,\mathbf{diag}(e^{\lambda_1}, \ldots, e^{\lambda_n})T^{-1} \end{aligned}$$

Analytic function of a matrix

for any analytic function $f : \mathbf{R} \to \mathbf{R}$, *i.e.*, given by power series

$$f(a) = \beta_0 + \beta_1 a + \beta_2 a^2 + \beta_3 a^3 + \cdots$$

we can define $f(A)$ for $A \in \mathbf{R}^{n \times n}$ (*i.e.*, overload f) as

$$f(A) = \beta_0 I + \beta_1 A + \beta_2 A^2 + \beta_3 A^3 + \cdots$$

substituting $A = T\Lambda T^{-1}$, we have

$$\begin{aligned} f(A) &= \beta_0 I + \beta_1 A + \beta_2 A^2 + \beta_3 A^3 + \cdots \\ &= \beta_0 T T^{-1} + \beta_1 T\Lambda T^{-1} + \beta_2 (T\Lambda T^{-1})^2 + \cdots \\ &= T\left(\beta_0 I + \beta_1 \Lambda + \beta_2 \Lambda^2 + \cdots\right) T^{-1} \\ &= T\,\mathbf{diag}(f(\lambda_1), \ldots, f(\lambda_n)) T^{-1} \end{aligned}$$

Solution via diagonalization

assume A is diagonalizable

consider LDS $\dot{x} = Ax$, with $T^{-1}AT = \Lambda$

then

$$\begin{aligned} x(t) &= e^{tA}x(0) \\ &= Te^{\Lambda t}T^{-1}x(0) \\ &= \sum_{i=1}^{n} e^{\lambda_i t}(w_i^T x(0))v_i \end{aligned}$$

thus: any trajectory can be expressed as linear combination of modes

interpretation:

- (left eigenvectors) decompose initial state $x(0)$ into modal components $w_i^T x(0)$
- $e^{\lambda_i t}$ term propagates ith mode forward t seconds
- reconstruct state as linear combination of (right) eigenvectors

application: for what $x(0)$ do we have $x(t) \to 0$ as $t \to \infty$?

divide eigenvalues into those with negative real parts

$$\Re\lambda_1 < 0, \ldots, \Re\lambda_s < 0,$$

and the others,

$$\Re\lambda_{s+1} \geq 0, \ldots, \Re\lambda_n \geq 0$$

from

$$x(t) = \sum_{i=1}^{n} e^{\lambda_i t}(w_i^T x(0)) v_i$$

condition for $x(t) \to 0$ is:

$$x(0) \in \operatorname{span}\{v_1, \ldots, v_s\},$$

or equivalently,

$$w_i^T x(0) = 0, \quad i = s+1, \ldots, n$$

(can you prove this?)

Stability of discrete-time systems

suppose A diagonalizable

consider discrete-time LDS $x(t+1) = Ax(t)$

if $A = T\Lambda T^{-1}$, then $A^k = T\Lambda^k T^{-1}$

then

$$x(t) = A^t x(0) = \sum_{i=1}^{n} \lambda_i^t (w_i^T x(0)) v_i \to 0 \quad \text{as } t \to \infty$$

for all $x(0)$ if and only if

$$|\lambda_i| < 1, \quad i = 1, \ldots, n.$$

we will see later that this is true even when A is not diagonalizable, so we have

fact: $x(t+1) = Ax(t)$ is stable if and only if all eigenvalues of A have magnitude less than one