

# Lecture 12

## Jordan canonical form

- Jordan canonical form
- generalized modes
- Cayley-Hamilton theorem

# Jordan canonical form

what if  $A$  cannot be diagonalized?

any matrix  $A \in \mathbf{R}^{n \times n}$  can be put in *Jordan canonical form* by a similarity transformation, *i.e.*

$$T^{-1}AT = J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{bmatrix}$$

where

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \in \mathbf{C}^{n_i \times n_i}$$

is called a *Jordan block* of size  $n_i$  with eigenvalue  $\lambda_i$  (so  $n = \sum_{i=1}^q n_i$ )

- $J$  is upper bidiagonal
- $J$  diagonal is the special case of  $n$  Jordan blocks of size  $n_i = 1$
- Jordan form is unique (up to permutations of the blocks)
- can have multiple blocks with same eigenvalue

**note:** JCF is a *conceptual tool*, never used in numerical computations!

$$\mathcal{X}(s) = \det(sI - A) = (s - \lambda_1)^{n_1} \cdots (s - \lambda_q)^{n_q}$$

hence distinct eigenvalues  $\Rightarrow n_i = 1 \Rightarrow A$  diagonalizable

$\dim \mathcal{N}(\lambda I - A)$  is the number of Jordan blocks with eigenvalue  $\lambda$

more generally,

$$\dim \mathcal{N}(\lambda I - A)^k = \sum_{\lambda_i = \lambda} \min\{k, n_i\}$$

so from  $\dim \mathcal{N}(\lambda I - A)^k$  for  $k = 1, 2, \dots$  we can determine the sizes of the Jordan blocks associated with  $\lambda$

- factor out  $T$  and  $T^{-1}$ ,  $\lambda I - A = T(\lambda I - J)T^{-1}$
- for, say, a block of size 3:

$$\lambda_i I - J_i = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad (\lambda_i I - J_i)^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\lambda_i I - J_i)^3 = 0$$

- for other blocks (say, size 3, for  $k \geq 2$ )

$$(\lambda_i I - J_j)^k = \begin{bmatrix} (\lambda_i - \lambda_j)^k & -k(\lambda_i - \lambda_j)^{k-1} & (k(k-1)/2)(\lambda_i - \lambda_j)^{k-2} \\ 0 & (\lambda_j - \lambda_i)^k & -k(\lambda_j - \lambda_i)^{k-1} \\ 0 & 0 & (\lambda_j - \lambda_i)^k \end{bmatrix}$$

## Generalized eigenvectors

suppose  $T^{-1}AT = J = \mathbf{diag}(J_1, \dots, J_q)$

express  $T$  as

$$T = [T_1 \ T_2 \ \cdots \ T_q]$$

where  $T_i \in \mathbf{C}^{n \times n_i}$  are the columns of  $T$  associated with  $i$ th Jordan block  $J_i$

we have  $AT_i = T_i J_i$

let  $T_i = [v_{i1} \ v_{i2} \ \cdots \ v_{in_i}]$

then we have:

$$Av_{i1} = \lambda_i v_{i1},$$

*i.e.*, the first column of each  $T_i$  is an eigenvector associated with e.v.  $\lambda_i$

for  $j = 2, \dots, n_i$ ,

$$Av_{ij} = v_{i \ j-1} + \lambda_i v_{ij}$$

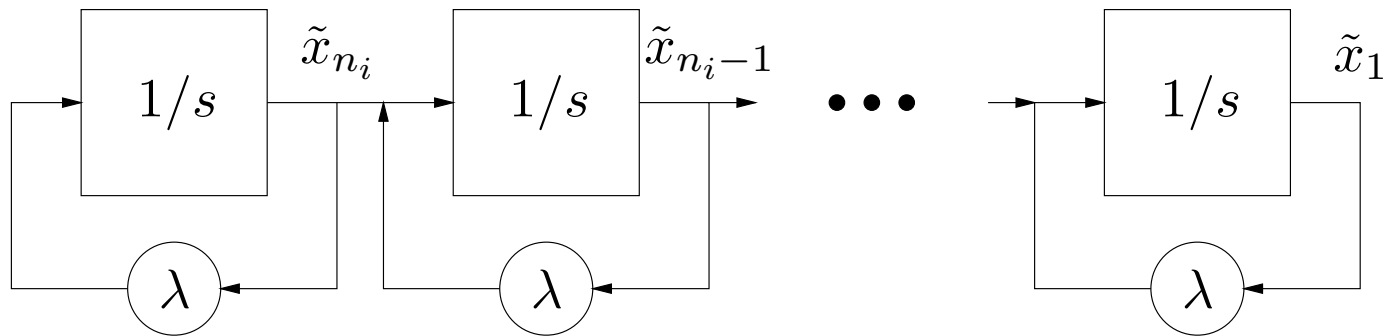
the vectors  $v_{i1}, \dots, v_{in_i}$  are sometimes called *generalized eigenvectors*

# Jordan form LDS

consider LDS  $\dot{x} = Ax$

by change of coordinates  $x = T\tilde{x}$ , can put into form  $\dot{\tilde{x}} = J\tilde{x}$

system is decomposed into independent 'Jordan block systems'  $\dot{\tilde{x}}_i = J_i\tilde{x}_i$



Jordan blocks are sometimes called Jordan chains

(block diagram shows why)

## Resolvent, exponential of Jordan block

resolvent of  $k \times k$  Jordan block with eigenvalue  $\lambda$ :

$$\begin{aligned}(sI - J_\lambda)^{-1} &= \begin{bmatrix} s - \lambda & -1 & & \\ & s - \lambda & \cdots & \\ & & \cdots & -1 \\ & & & s - \lambda \end{bmatrix}^{-1} \\ &= \begin{bmatrix} (s - \lambda)^{-1} & (s - \lambda)^{-2} & \cdots & (s - \lambda)^{-k} \\ & (s - \lambda)^{-1} & \cdots & (s - \lambda)^{-k+1} \\ & & \cdots & \vdots \\ & & & (s - \lambda)^{-1} \end{bmatrix} \\ &= (s - \lambda)^{-1}I + (s - \lambda)^{-2}F_1 + \cdots + (s - \lambda)^{-k}F_{k-1}\end{aligned}$$

where  $F_i$  is the matrix with ones on the  $i$ th upper diagonal

by inverse Laplace transform, exponential is:

$$\begin{aligned} e^{tJ_\lambda} &= e^{t\lambda} (I + tF_1 + \cdots + (t^{k-1}/(k-1)!)F_{k-1}) \\ &= e^{t\lambda} \begin{bmatrix} 1 & t & \cdots & t^{k-1}/(k-1)! \\ & 1 & \cdots & t^{k-2}/(k-2)! \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix} \end{aligned}$$

Jordan blocks yield:

- repeated poles in resolvent
- terms of form  $t^p e^{t\lambda}$  in  $e^{tA}$

# Generalized modes

consider  $\dot{x} = Ax$ , with

$$x(0) = a_1 v_{i1} + \cdots + a_{n_i} v_{in_i} = T_i a$$

then  $x(t) = T e^{Jt} \tilde{x}(0) = T_i e^{J_i t} a$

- trajectory stays in span of generalized eigenvectors
- coefficients have form  $p(t)e^{\lambda t}$ , where  $p$  is polynomial
- such solutions are called *generalized modes* of the system

with general  $x(0)$  we can write

$$x(t) = e^{tA}x(0) = Te^{tJ}T^{-1}x(0) = \sum_{i=1}^q T_i e^{tJ_i} (S_i^T x(0))$$

where

$$T^{-1} = \begin{bmatrix} S_1^T \\ \vdots \\ S_q^T \end{bmatrix}$$

hence: all solutions of  $\dot{x} = Ax$  are linear combinations of (generalized) modes

# Cayley-Hamilton theorem

if  $p(s) = a_0 + a_1s + \cdots + a_k s^k$  is a polynomial and  $A \in \mathbf{R}^{n \times n}$ , we define

$$p(A) = a_0I + a_1A + \cdots + a_k A^k$$

**Cayley-Hamilton theorem:** for any  $A \in \mathbf{R}^{n \times n}$  we have  $\mathcal{X}(A) = 0$ , where  $\mathcal{X}(s) = \det(sI - A)$

**example:** with  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$  we have  $\mathcal{X}(s) = s^2 - 5s - 2$ , so

$$\begin{aligned} \mathcal{X}(A) &= A^2 - 5A - 2I \\ &= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - 5 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - 2I \\ &= 0 \end{aligned}$$

**corollary:** for every  $p \in \mathbf{Z}_+$ , we have

$$A^p \in \text{span} \{ I, A, A^2, \dots, A^{n-1} \}$$

(and if  $A$  is invertible, also for  $p \in \mathbf{Z}$ )

*i.e.*, every power of  $A$  can be expressed as linear combination of  $I, A, \dots, A^{n-1}$

**proof:** divide  $\mathcal{X}(s)$  into  $s^p$  to get  $s^p = q(s)\mathcal{X}(s) + r(s)$

$r = \alpha_0 + \alpha_1 s + \dots + \alpha_{n-1} s^{n-1}$  is remainder polynomial

then

$$A^p = q(A)\mathcal{X}(A) + r(A) = r(A) = \alpha_0 I + \alpha_1 A + \dots + \alpha_{n-1} A^{n-1}$$

for  $p = -1$ : rewrite C-H theorem

$$\mathcal{X}(A) = A^n + a_{n-1}A^{n-1} + \cdots + a_0I = 0$$

as

$$I = A \left( -(a_1/a_0)I - (a_2/a_0)A - \cdots - (1/a_0)A^{n-1} \right)$$

( $A$  is invertible  $\Leftrightarrow a_0 \neq 0$ ) so

$$A^{-1} = -(a_1/a_0)I - (a_2/a_0)A - \cdots - (1/a_0)A^{n-1}$$

*i.e.*, inverse is linear combination of  $A^k$ ,  $k = 0, \dots, n-1$

# Proof of C-H theorem

first assume  $A$  is diagonalizable:  $T^{-1}AT = \Lambda$

$$\mathcal{X}(s) = (s - \lambda_1) \cdots (s - \lambda_n)$$

since

$$\mathcal{X}(A) = \mathcal{X}(T\Lambda T^{-1}) = T\mathcal{X}(\Lambda)T^{-1}$$

it suffices to show  $\mathcal{X}(\Lambda) = 0$

$$\begin{aligned}\mathcal{X}(\Lambda) &= (\Lambda - \lambda_1 I) \cdots (\Lambda - \lambda_n I) \\ &= \mathbf{diag}(0, \lambda_2 - \lambda_1, \dots, \lambda_n - \lambda_1) \cdots \mathbf{diag}(\lambda_1 - \lambda_n, \dots, \lambda_{n-1} - \lambda_n, 0) \\ &= 0\end{aligned}$$

now let's do general case:  $T^{-1}AT = J$

$$\mathcal{X}(s) = (s - \lambda_1)^{n_1} \cdots (s - \lambda_q)^{n_q}$$

suffices to show  $\mathcal{X}(J_i) = 0$

$$\mathcal{X}(J_i) = (J_i - \lambda_1 I)^{n_1} \cdots \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ & & & \ddots \end{bmatrix}^{n_i}}_{(J_i - \lambda_i I)^{n_i}} \cdots (J_i - \lambda_q I)^{n_q} = 0$$