Lecture 2
Linear functions and examples

- linear equations and functions
- engineering examples
- interpretations

Linear equations

consider system of linear equations

$$\begin{array}{rcrcrcr}
y_1 & = & a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n \\
y_2 & = & a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n \\
 & \vdots & & & & & \\
y_m & = & a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n
\end{array}$$

can be written in matrix form as $y = Ax$, where

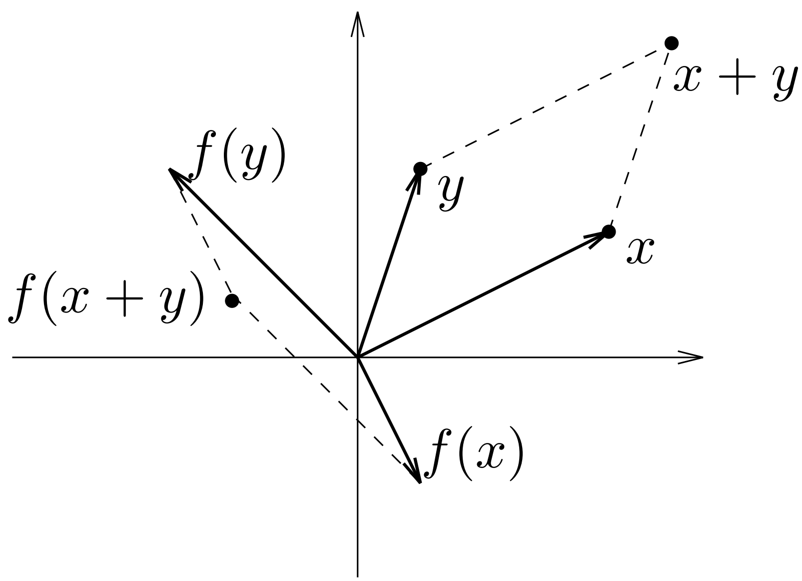
$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Linear functions

a function $f : \mathbf{R}^n \longrightarrow \mathbf{R}^m$ is *linear* if

- $f(x+y) = f(x) + f(y)$, $\forall x, y \in \mathbf{R}^n$
- $f(\alpha x) = \alpha f(x)$, $\forall x \in \mathbf{R}^n$ $\forall \alpha \in \mathbf{R}$

$i.e.$, *superposition* holds



Matrix multiplication function

- consider function $f : \mathbf{R}^n \to \mathbf{R}^m$ given by $f(x) = Ax$, where $A \in \mathbf{R}^{m \times n}$

- matrix multiplication function f is linear

- **converse** is true: **any** linear function $f : \mathbf{R}^n \to \mathbf{R}^m$ can be written as $f(x) = Ax$ for some $A \in \mathbf{R}^{m \times n}$

- representation via matrix multiplication is unique: for any linear function f there is only one matrix A for which $f(x) = Ax$ for all x

- $y = Ax$ is a concrete representation of a generic linear function

Interpretations of $y = Ax$

- y is measurement or observation; x is unknown to be determined
- x is 'input' or 'action'; y is 'output' or 'result'
- $y = Ax$ defines a function or transformation that maps $x \in \mathbf{R}^n$ into $y \in \mathbf{R}^m$

Interpretation of a_{ij}

$$y_i = \sum_{j=1}^{n} a_{ij} x_j$$

a_{ij} is *gain factor* from jth input (x_j) to ith output (y_i)

thus, $e.g.$,

- ith *row* of A concerns ith *output*
- jth *column* of A concerns jth *input*
- $a_{27} = 0$ means 2nd output (y_2) doesn't depend on 7th input (x_7)
- $|a_{31}| \gg |a_{3j}|$ for $j \neq 1$ means y_3 depends mainly on x_1

- $|a_{52}| \gg |a_{i2}|$ for $i \neq 5$ means x_2 affects mainly y_5

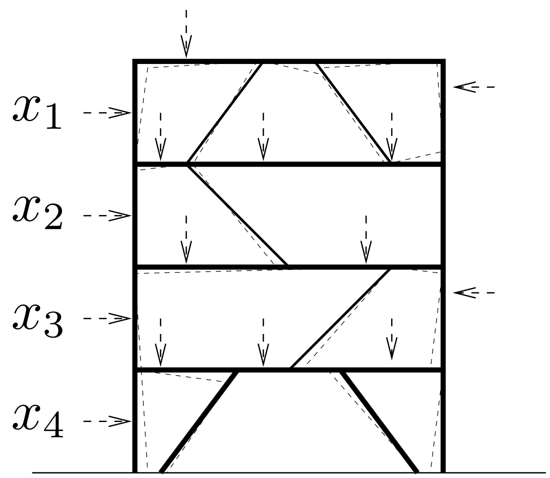
- A is lower triangular, *i.e.*, $a_{ij} = 0$ for $i < j$, means y_i only depends on $x_1, \ldots, x_i$

- A is diagonal, *i.e.*, $a_{ij} = 0$ for $i \neq j$, means ith output depends only on ith input

more generally, **sparsity pattern** of A, *i.e.*, list of zero/nonzero entries of A, shows which x_j affect which y_i

Linear elastic structure

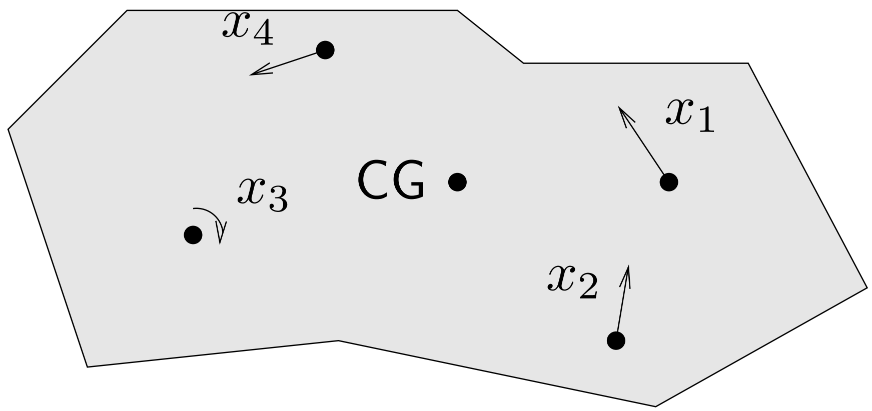
- x_j is external force applied at some node, in some fixed direction
- y_i is (small) deflection of some node, in some fixed direction



(provided x, y are small) we have $y \approx Ax$

- A is called the *compliance matrix*
- a_{ij} gives deflection i per unit force at j (in m/N)

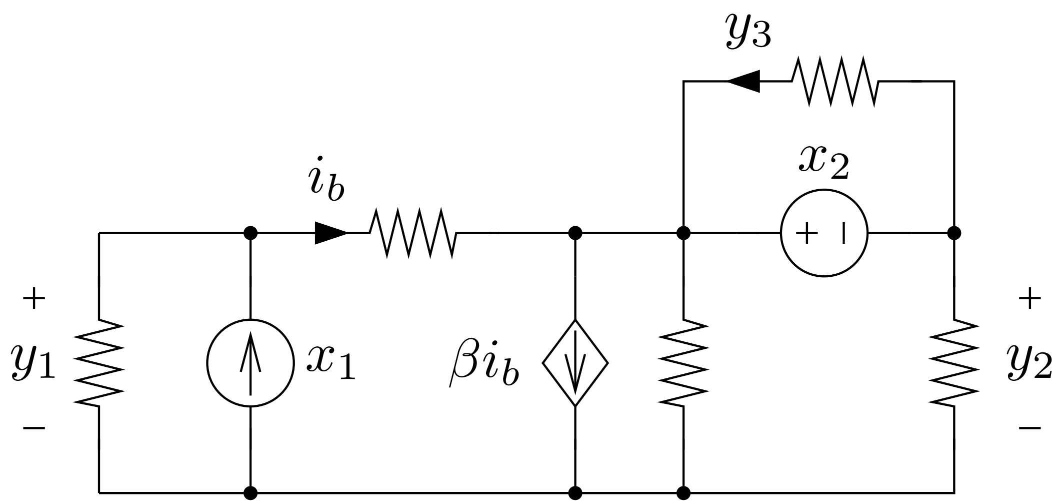
Total force/torque on rigid body



- x_j is external force/torque applied at some point/direction/axis
- $y \in \mathbf{R}^6$ is resulting total force & torque on body
  ($y_1,\ y_2,\ y_3$ are $\mathbf{x}$-, $\mathbf{y}$-, $\mathbf{z}$- components of total force,
  $y_4,\ y_5,\ y_6$ are $\mathbf{x}$-, $\mathbf{y}$-, $\mathbf{z}$- components of total torque)
- we have $y = Ax$
- A depends on geometry
  (of applied forces and torques with respect to center of gravity CG)
- jth column gives resulting force & torque for unit force/torque j

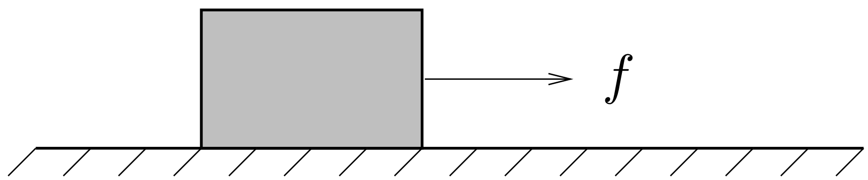
Linear static circuit

interconnection of resistors, linear dependent (controlled) sources, and independent sources



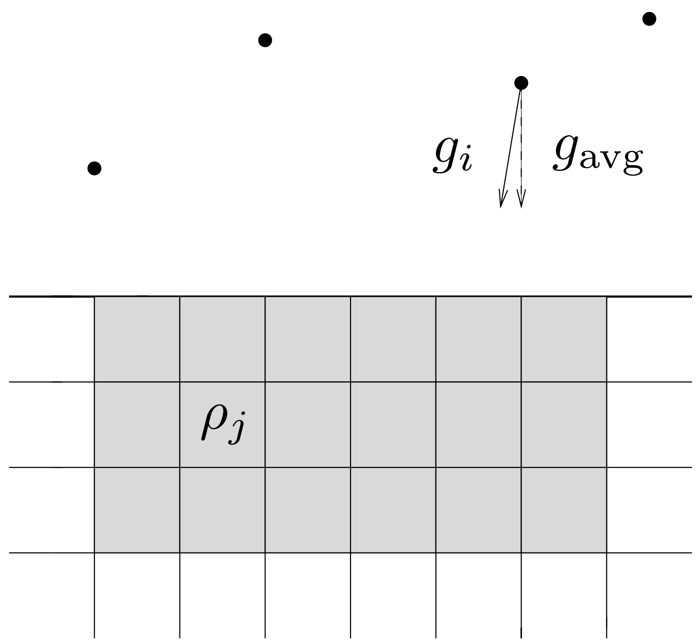
- x_j is value of independent source j
- y_i is some circuit variable (voltage, current)
- we have $y = Ax$
- if x_j are currents and y_i are voltages, A is called the *impedance* or *resistance* matrix

Final position/velocity of mass due to applied forces



- unit mass, zero position/velocity at $t = 0$, subject to force $f(t)$ for $0 \leq t \leq n$
- $f(t) = x_j$ for $j - 1 \leq t < j$, $j = 1, \dots, n$
  (x is the sequence of applied forces, constant in each interval)
- y_1, y_2 are final position and velocity ($i.e.$, at $t = n$)
- we have $y = Ax$
- a_{1j} gives influence of applied force during $j - 1 \leq t < j$ on final position
- a_{2j} gives influence of applied force during $j - 1 \leq t < j$ on final velocity

Gravimeter prospecting



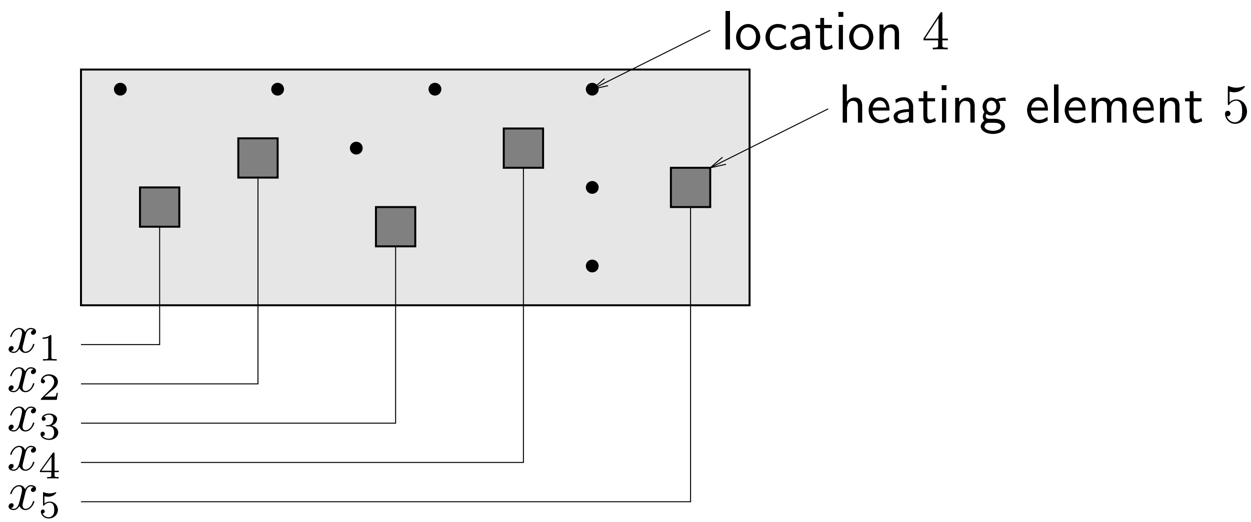
- $x_j = \rho_j - \rho_{\text{avg}}$ is (excess) mass density of earth in voxel j;
- y_i is measured *gravity anomaly* at location i, $i.e.$, some component (typically vertical) of $g_i - g_{\text{avg}}$
- $y = Ax$

- A comes from physics and geometry

- jth column of A shows sensor readings caused by unit density anomaly at voxel j

- ith row of A shows sensitivity pattern of sensor i

Thermal system



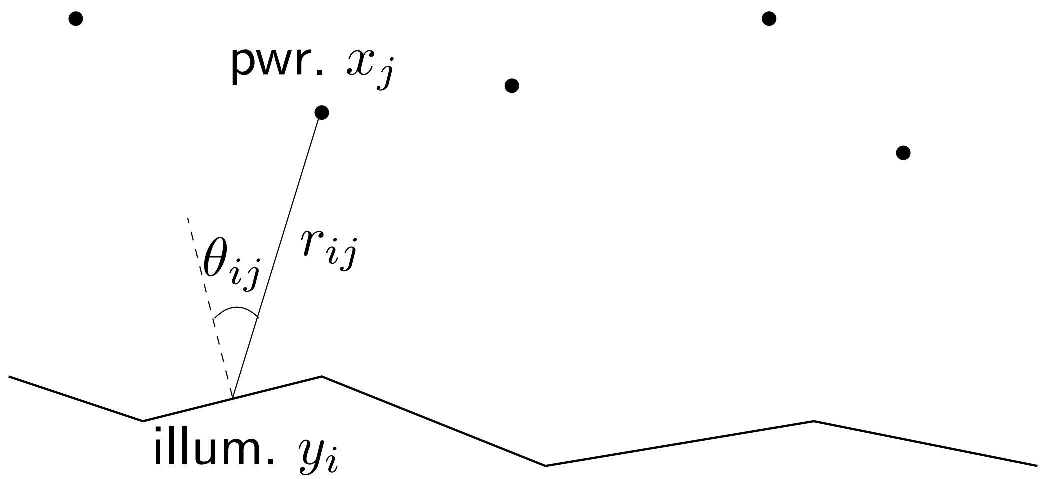
- x_j is power of jth heating element or heat source
- y_i is change in steady-state temperature at location i
- thermal transport via conduction
- $y = Ax$

- a_{ij} gives influence of heater j at location i (in °C/W)

- jth column of A gives pattern of steady-state temperature rise due to 1W at heater j

- ith row shows how heaters affect location i

Illumination with multiple lamps



- n lamps illuminating m (small, flat) patches, no shadows
- x_j is power of jth lamp; y_i is illumination level of patch i
- $y = Ax$, where $a_{ij} = r_{ij}^{-2} \max\{\cos\theta_{ij}, 0\}$

  ($\cos\theta_{ij} < 0$ means patch i is shaded from lamp j)
- jth column of A shows illumination pattern from lamp j

Signal and interference power in wireless system

- n transmitter/receiver pairs
- transmitter j transmits to receiver j (and, inadvertantly, to the other receivers)
- p_j is power of jth transmitter
- s_i is received signal power of ith receiver
- z_i is received interference power of ith receiver
- G_{ij} is path gain from transmitter j to receiver i
- we have $s = Ap$, $z = Bp$, where

$$a_{ij} = \begin{cases} G_{ii} & i = j \\ 0 & i \neq j \end{cases} \qquad b_{ij} = \begin{cases} 0 & i = j \\ G_{ij} & i \neq j \end{cases}$$

- A is diagonal; B has zero diagonal (ideally, A is 'large', B is 'small')

Cost of production

production *inputs* (materials, parts, labor, . . .) are combined to make a number of *products*

- x_j is price per unit of production input j
- a_{ij} is units of production input j required to manufacture one unit of product i
- y_i is production cost per unit of product i
- we have $y = Ax$
- ith row of A is *bill of materials* for unit of product i

production inputs needed

- q_i is quantity of product i to be produced
- r_j is total quantity of production input j needed
- we have $r = A^T q$

total production cost is

$$r^T x = (A^T q)^T x = q^T A x$$

Network traffic and flows

- n flows with rates $f_1, \ldots, f_n$ pass from their source nodes to their destination nodes over fixed routes in a network

- t_i, traffic on link i, is sum of rates of flows passing through it

- flow routes given by *flow-link incidence matrix*

$$A_{ij} = \begin{cases} 1 & \text{flow } j \text{ goes over link } i \\ 0 & \text{otherwise} \end{cases}$$

- traffic and flow rates related by $t = Af$

link delays and flow latency

- let $d_1, \ldots, d_m$ be link delays, and $l_1, \ldots, l_n$ be latency (total travel time) of flows
- $l = A^T d$
- $f^T l = f^T A^T d = (Af)^T d = t^T d$, total # of packets in network

Linearization

- if $f : \mathbf{R}^n \to \mathbf{R}^m$ is differentiable at $x_0 \in \mathbf{R}^n$, then

$$x \text{ near } x_0 \Longrightarrow f(x) \text{ very near } f(x_0) + Df(x_0)(x - x_0)$$

  where
$$Df(x_0)_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{x_0}$$
  is derivative (Jacobian) matrix

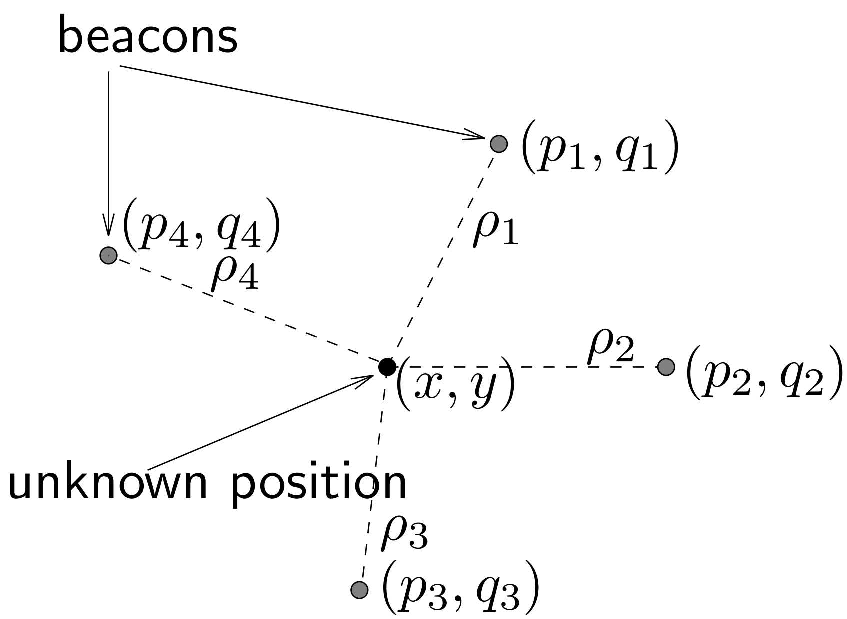
- with $y = f(x)$, $y_0 = f(x_0)$, define *input deviation* $\delta x := x - x_0$, *output deviation* $\delta y := y - y_0$

- then we have $\delta y \approx Df(x_0)\delta x$

- when deviations are small, they are (approximately) related by a linear function

Navigation by range measurement

- (x, y) unknown coordinates in plane
- (p_i, q_i) known coordinates of beacons for $i = 1, 2, 3, 4$
- ρ_i measured (known) distance or range from beacon i



- $\rho \in \mathbf{R}^4$ is a nonlinear function of $(x, y) \in \mathbf{R}^2$:

$$\rho_i(x, y) = \sqrt{(x - p_i)^2 + (y - q_i)^2}$$

- linearize around (x_0, y_0): $\delta\rho \approx A \begin{bmatrix} \delta x \\ \delta y \end{bmatrix}$, where

$$a_{i1} = \frac{(x_0 - p_i)}{\sqrt{(x_0 - p_i)^2 + (y_0 - q_i)^2}}, \quad a_{i2} = \frac{(y_0 - q_i)}{\sqrt{(x_0 - p_i)^2 + (y_0 - q_i)^2}}$$

- ith row of A shows (approximate) change in ith range measurement for (small) shift in (x, y) from (x_0, y_0)

- first column of A shows sensitivity of range measurements to (small) change in x from x_0

- obvious application: (x_0, y_0) is last navigation fix; (x, y) is current position, a short time later

Broad categories of applications

linear model or function $y = Ax$

some broad categories of applications:

- estimation or inversion
- control or design
- mapping or transformation

(this list is not exclusive; can have combinations . . .)

Estimation or inversion

$$y = Ax$$

- y_i is ith measurement or sensor reading (which we know)
- x_j is jth parameter to be estimated or determined
- a_{ij} is sensitivity of ith sensor to jth parameter

sample problems:

- find x, given y
- find all x's that result in y ($i.e.$, all x's consistent with measurements)
- if there is no x such that $y = Ax$, find x s.t. $y \approx Ax$ ($i.e.$, if the sensor readings are inconsistent, find x which is almost consistent)

Control or design

$$y = Ax$$

- x is vector of design parameters or inputs (which we can choose)
- y is vector of results, or outcomes
- A describes how input choices affect results

sample problems:

- find x so that $y = y_{\rm des}$
- find all x's that result in $y = y_{\rm des}$ (*i.e.*, find all designs that meet specifications)
- among x's that satisfy $y = y_{\rm des}$, find a small one (*i.e.*, find a small or efficient x that meets specifications)

Mapping or transformation

- x is mapped or transformed to y by linear function $y = Ax$

sample problems:

- determine if there is an x that maps to a given y
- (if possible) find *an* x that maps to y
- find *all* x's that map to a given y
- if there is only one x that maps to y, find it ($i.e.$, decode or undo the mapping)

Matrix multiplication as mixture of columns

write $A \in \mathbf{R}^{m \times n}$ in terms of its columns:

$$A = \left[\begin{array}{cccc} a_1 & a_2 & \cdots & a_n \end{array} \right]$$

where $a_j \in \mathbf{R}^m$

then $y = Ax$ can be written as

$$y = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

(x_j's are scalars, a_j's are m-vectors)

- y is a (linear) combination or mixture of the columns of A
- coefficients of x give coefficients of mixture

an important example: $x = e_j$, the jth *unit vector*

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

then $Ae_j = a_j$, the jth column of A

(e_j corresponds to a pure mixture, giving only column j)

Matrix multiplication as inner product with rows

write A in terms of its rows:

$$A = \begin{bmatrix} \tilde{a}_1^T \\ \tilde{a}_2^T \\ \vdots \\ \tilde{a}_m^T \end{bmatrix}$$

where $\tilde{a}_i \in \mathbf{R}^n$

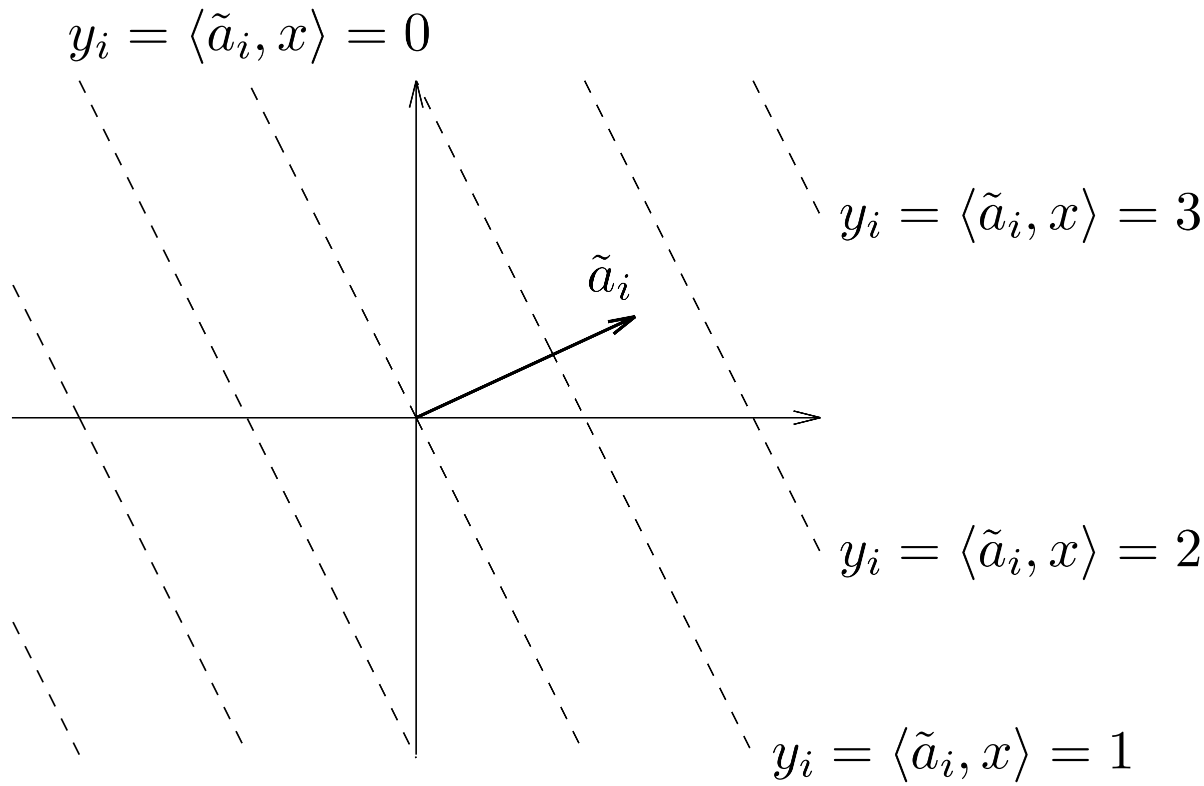
then $y = Ax$ can be written as

$$y = \begin{bmatrix} \tilde{a}_1^T x \\ \tilde{a}_2^T x \\ \vdots \\ \tilde{a}_m^T x \end{bmatrix}$$

thus $y_i = \langle \tilde{a}_i, x \rangle$, *i.e.*, y_i is inner product of ith row of A with x

geometric interpretation:

$y_i = \tilde{a}_i^T x = \alpha$ is a hyperplane in $\mathbf{R}^n$ (normal to $\tilde{a}_i$)



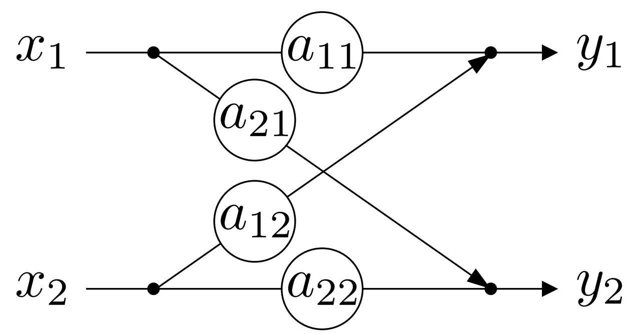
Block diagram representation

$y = Ax$ can be represented by a *signal flow graph* or *block diagram*

$e.g.$ for $m = n = 2$, we represent

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

as



- a_{ij} is the gain along the path from jth input to ith output
- (by not drawing paths with zero gain) shows sparsity structure of A ($e.g.$, diagonal, block upper triangular, arrow . . .)

example: block upper triangular, $i.e.$,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where $A_{11} \in \mathbf{R}^{m_1 \times n_1}$, $A_{12} \in \mathbf{R}^{m_1 \times n_2}$, $A_{21} \in \mathbf{R}^{m_2 \times n_1}$, $A_{22} \in \mathbf{R}^{m_2 \times n_2}$

partition x and y conformably as

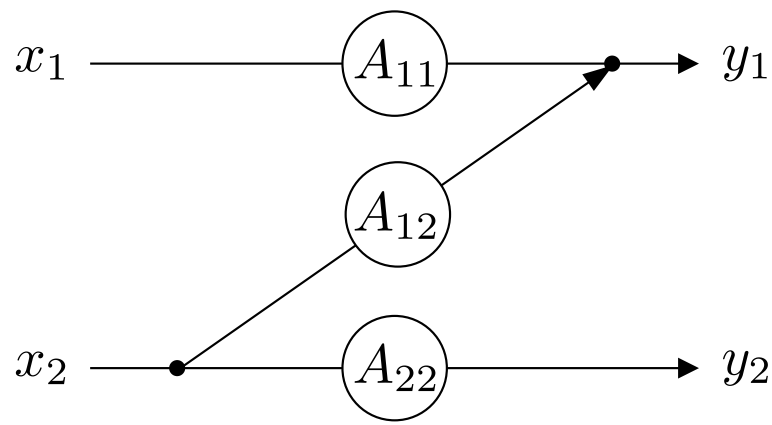
$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

($x_1 \in \mathbf{R}^{n_1}$, $x_2 \in \mathbf{R}^{n_2}$, $y_1 \in \mathbf{R}^{m_1}$, $y_2 \in \mathbf{R}^{m_2}$) so

$$y_1 = A_{11}x_1 + A_{12}x_2, \qquad y_2 = A_{22}x_2,$$

$i.e.$, y_2 doesn't depend on x_1

block diagram:



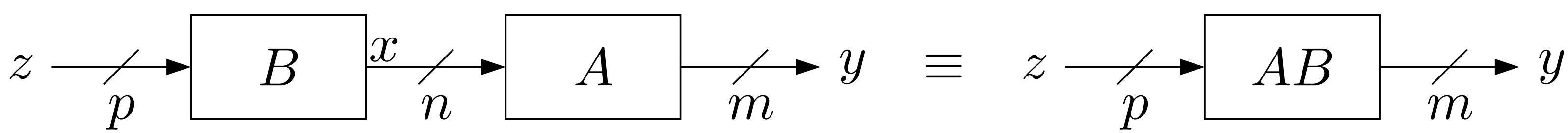
. . . no path from x_1 to y_2, so y_2 doesn't depend on x_1

Matrix multiplication as composition

for $A \in \mathbf{R}^{m\times n}$ and $B \in \mathbf{R}^{n\times p}$, $C = AB \in \mathbf{R}^{m\times p}$ where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

composition interpretation: $y = Cz$ represents composition of $y = Ax$ and $x = Bz$



(note that B is on left in block diagram)

Column and row interpretations

can write product $C = AB$ as

$$C = \left[\begin{array}{ccc} c_1 & \cdots & c_p \end{array}\right] = AB = \left[\begin{array}{ccc} Ab_1 & \cdots & Ab_p \end{array}\right]$$

$i.e.$, ith column of C is A acting on ith column of B

similarly we can write

$$C = \left[\begin{array}{c} \tilde{c}_1^T \\ \vdots \\ \tilde{c}_m^T \end{array}\right] = AB = \left[\begin{array}{c} \tilde{a}_1^T B \\ \vdots \\ \tilde{a}_m^T B \end{array}\right]$$

$i.e.$, ith row of C is ith row of A acting (on left) on B

Inner product interpretation

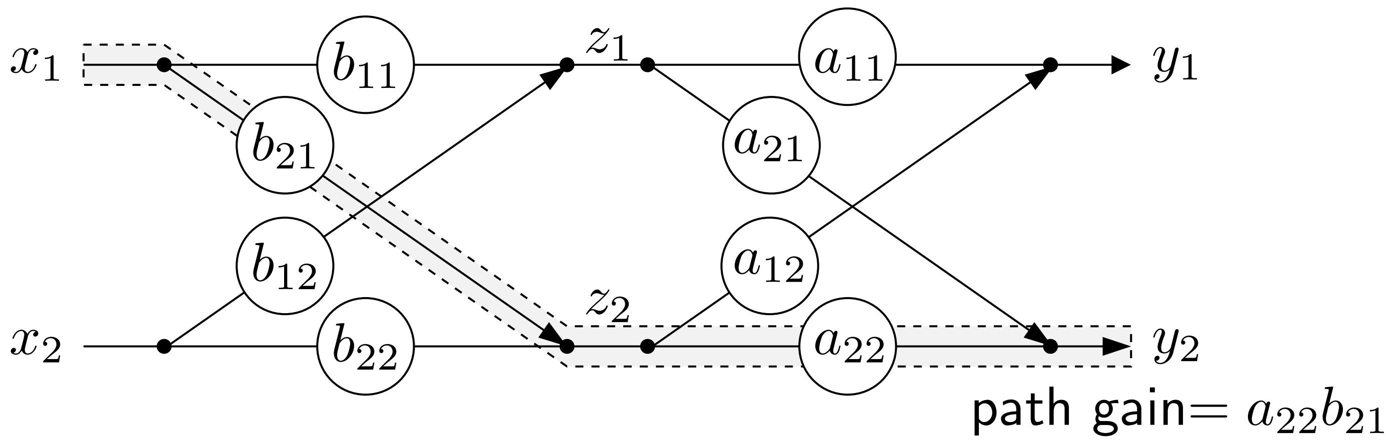
inner product interpretation:

$$c_{ij} = \tilde{a}_i^T b_j = \langle \tilde{a}_i, b_j \rangle$$

$i.e.$, entries of C are inner products of rows of A and columns of B

- $c_{ij} = 0$ means ith row of A is orthogonal to jth column of B
- **Gram matrix** of vectors $f_1, \ldots, f_n$ defined as $G_{ij} = f_i^T f_j$ (gives inner product of each vector with the others)
- $G = [f_1 \ \cdots \ f_n]^T [f_1 \ \cdots \ f_n]$

Matrix multiplication interpretation via paths



- $a_{ik}b_{kj}$ is gain of path from input j to output i via k

- c_{ij} is sum of gains over *all* paths from input j to output i