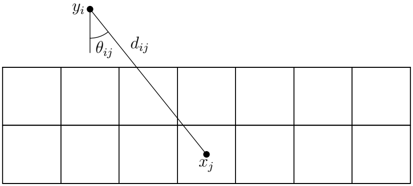
Gravimetric prospecting



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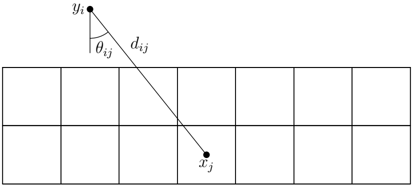

- x_j is excess density of voxel j

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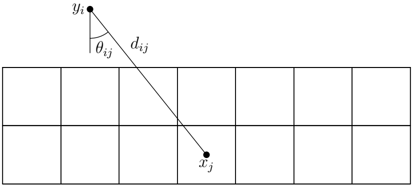

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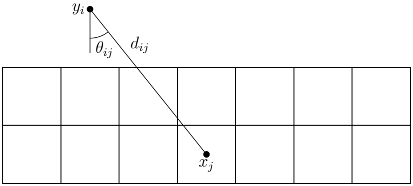

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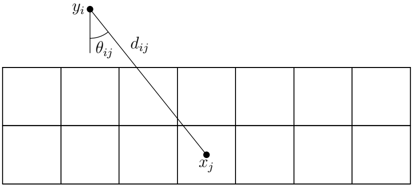

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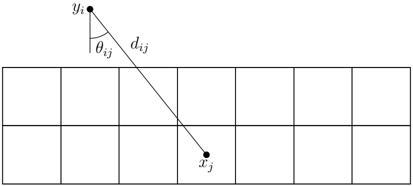
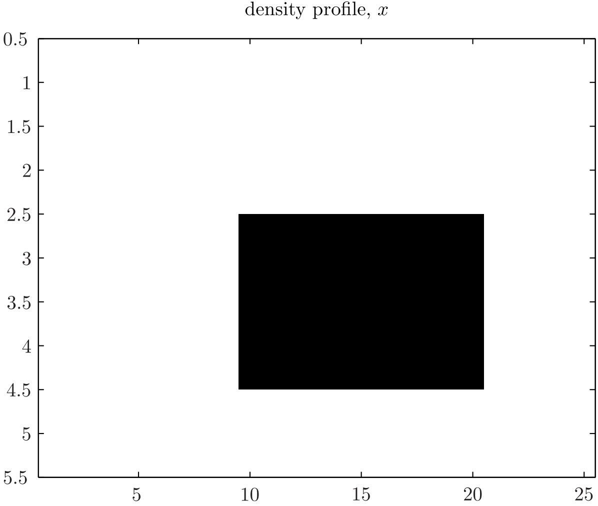


- x_j is excess density of voxel j
- y_i is gravity anomaly at location i
- θ_{ij} is angle from location i to voxel j
- d_{ij} is distance from location i to voxel j
- Newton's law of gravitation:

$$y_i = \sum_{j=1}^{n} \frac{G\cos(\theta_{ij})}{d_{ij}^2} x_j, \qquad i = 1, \ldots, m$$

Gravimetric prospecting

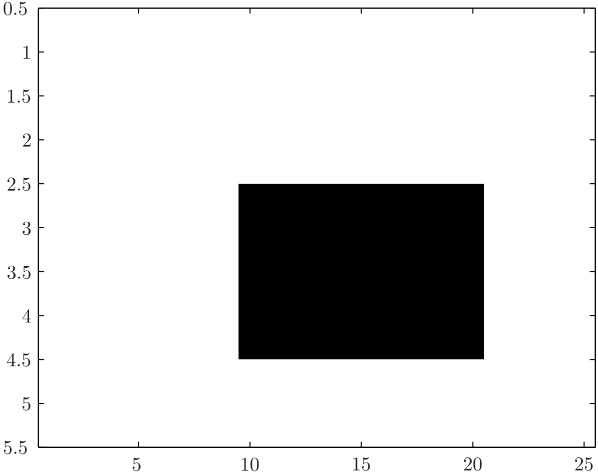
an example



Gravimetric prospecting

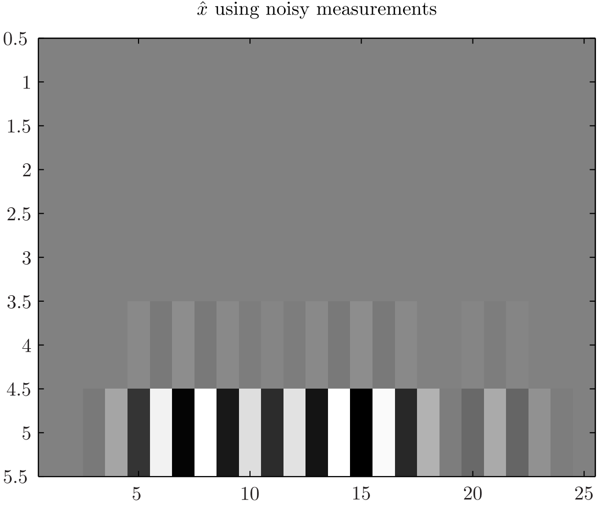
estimated density with exact measurements

$\hat{x}$ using exact measurements



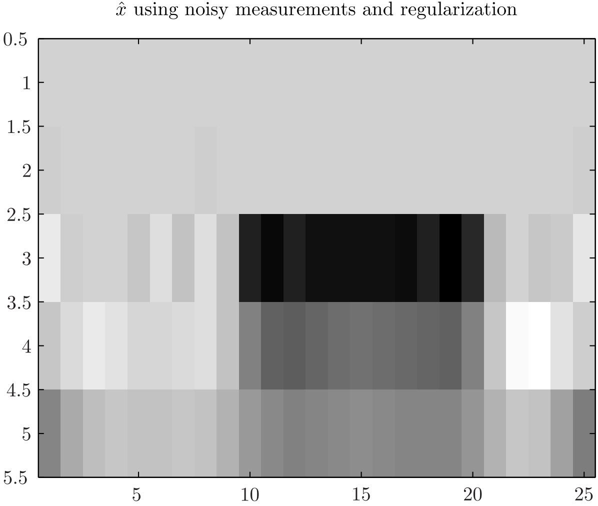
Gravimetric prospecting

estimated density with noisy measurements ($\pm 0.01\%$)

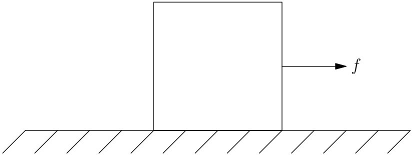


Gravimetric prospecting

estimated density with noisy measurements and regularization



Forces applied to a unit mass



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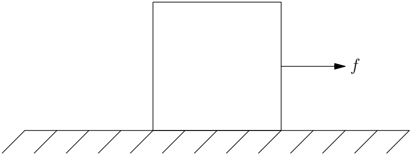

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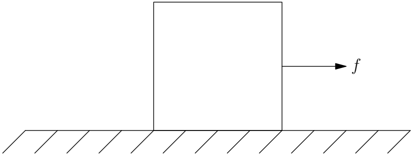

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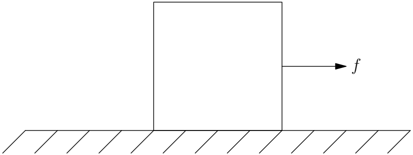

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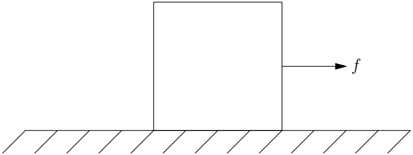

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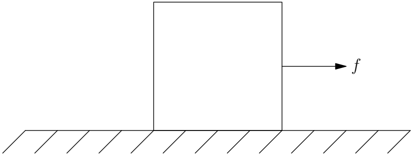

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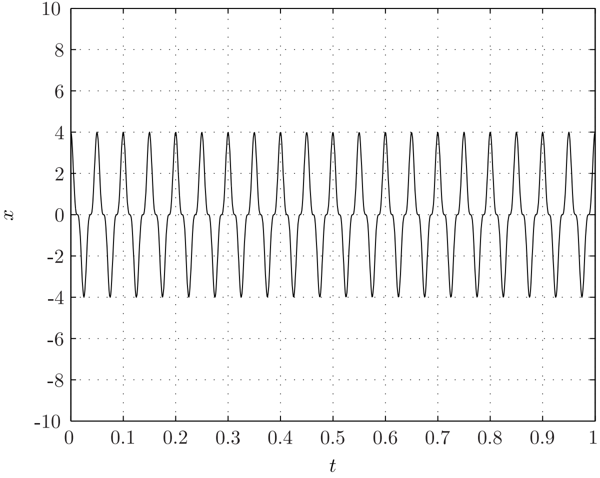
- final velocity:

$$y_2 = x_1 + x_2 + \cdots + x_n$$

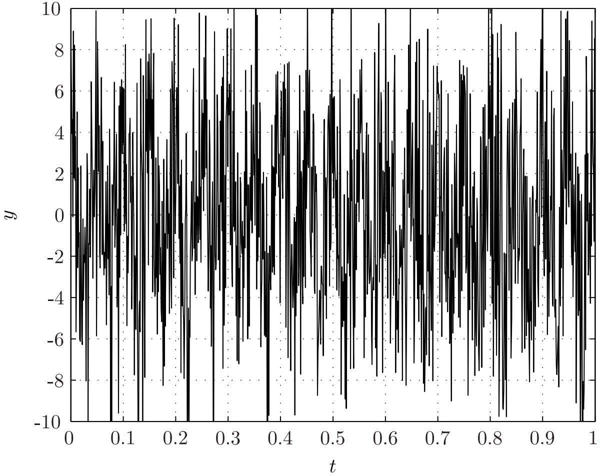
A noisy signal

a sum of two sinusoids



A noisy signal

a sum of two sinusoids corrupted by noise



Approximate Fourier transform

$$X(f) = \int_{-\infty}^{+\infty} x(t) e^{-2\pi i f t}\, dt$$

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&\approx \frac{1}{N} \sum_{n=0}^{N-1} x\left(\frac{n}{N}\right) e^{-2\pi i f n/N}
\end{aligned}
$$

The discrete Fourier transform

define the discrete Fourier transform of a signal $x_0, \ldots, x_{N-1}$:

$$X_k = \sum_{n=0}^{N-1} x_n e^{-2\pi i k n/N}, \qquad k = 0, \ldots, N-1$$

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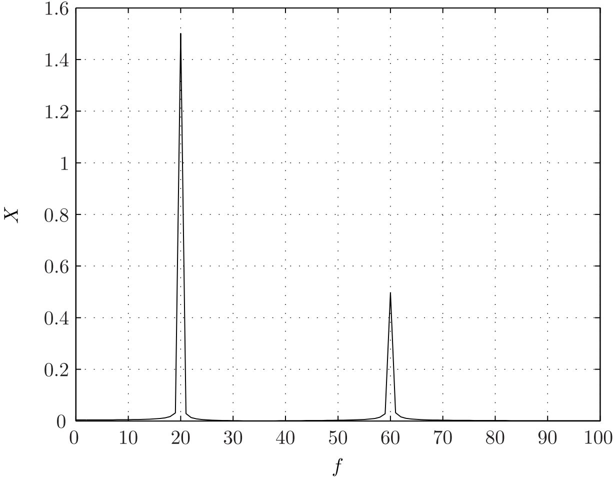
we can also express these equations as

$$\begin{aligned} X_0 &= \quad e^{-2\pi i(0)(0)/N} \quad x_0 + \cdots + \quad e^{-2\pi i(0)(N-1)/N} \quad x_{N-1} \\ &\vdots \\ X_k &= e^{-2\pi i(N-1)(0)/N} x_0 + \cdots + e^{-2\pi i(N-1)(N-1)/N} x_{N-1} \end{aligned}$$

Filtering in the frequency domain

DFT of the uncorrupted signal



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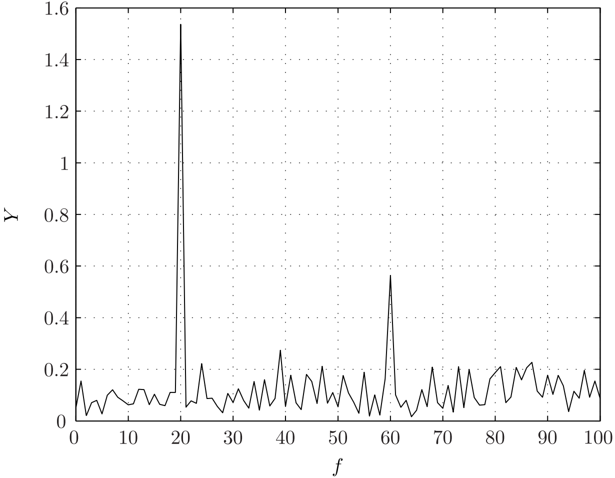
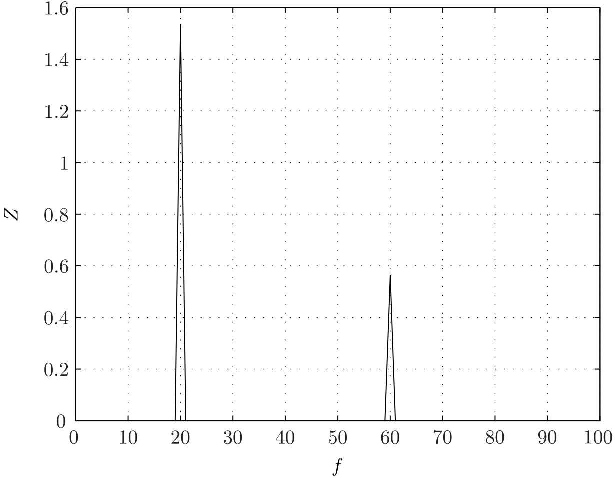


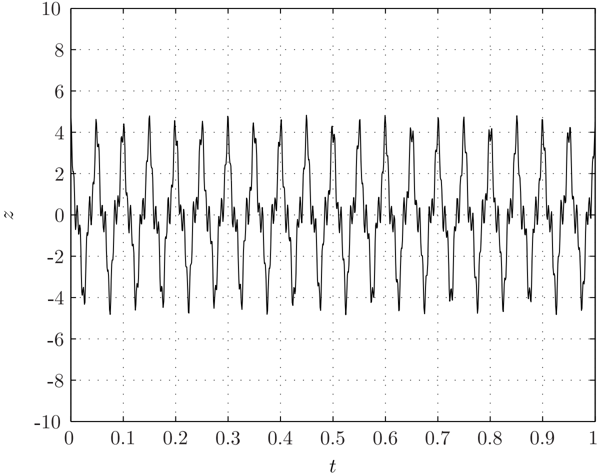
Filtering in the frequency domain

apply threshold filter



The denoised signal

applying the inverse DFT gives



Systems of linear equations

system of linear equations:

$$\begin{array}{ccccccccc} y_1 & = & A_{11} & x_1 & + & \cdots & + & A_{1n} & x_n \\ & \vdots & & & & & & & \\ y_m & = & A_{m1} & x_1 & + & \cdots & + & A_{mn} & x_n \end{array}$$

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where

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}, \qquad A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

right side of system *defines* matrix-vector multiplication

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- given y_{des}, find *an* x such that $Ax = y_{\text{des}}$
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- given y_{des}, find the “smallest” x such that $Ax = y_{\text{des}}$

Standard basis vectors

- jth standard basis vector in $\mathbb{R}^n$ is vector $e_j \in \mathbb{R}^n$ such that

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- `MATLAB`: `sparse(j,1,1,n,1)`

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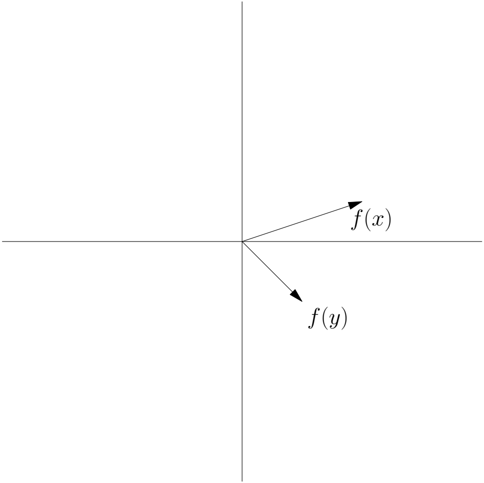
- *additive:* $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}^n$
- *homogeneous:* $f(\alpha x) = \alpha f(x)$ for all $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^n$

Superposition principle



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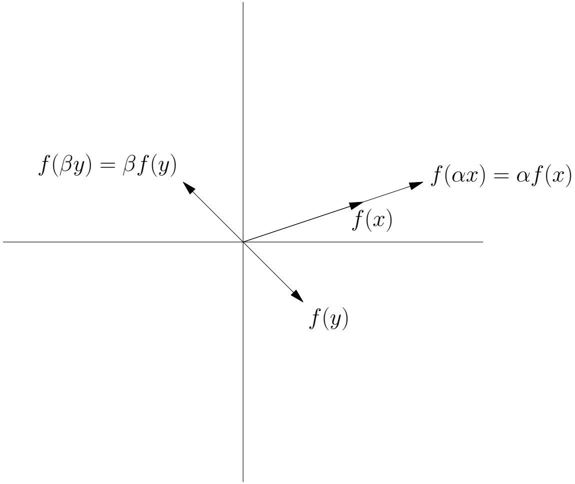

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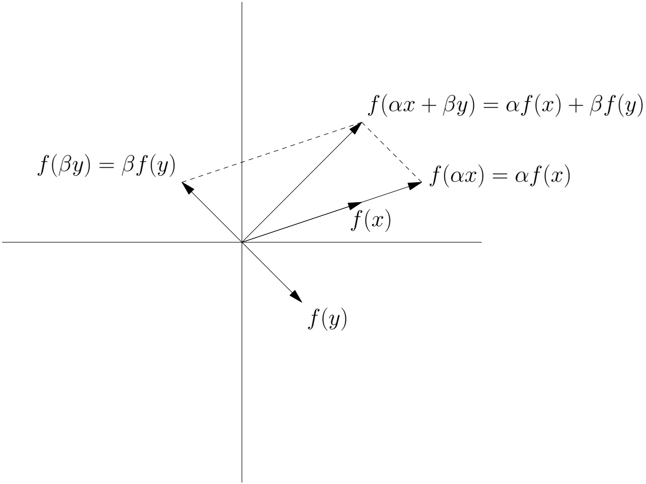

Matrix multiplication function

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- matrix is concrete representation of abstract linear function

Linearization

if $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x_0 \in \mathbb{R}^n$, then

$$f(x) \text{ is very near } f(x_0) + Df(x_0)(x - x_0)$$

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where

$$\left[Df(x_0)\right]_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{x=x_0}$$

is the derivative (Jacobian) matrix

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define the deviations

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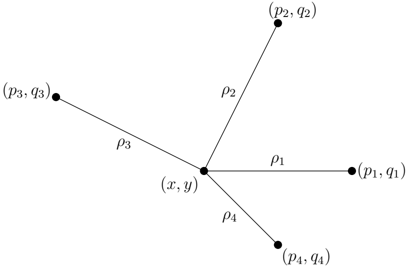
small deviations are (approximately) related by a linear function:

$$\delta y \approx Df(x_0)\delta x$$

Linearized range measurements



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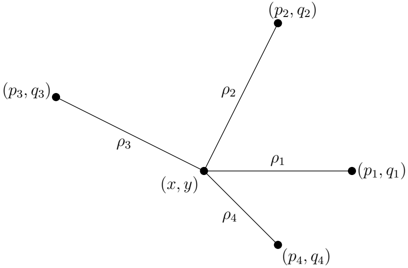

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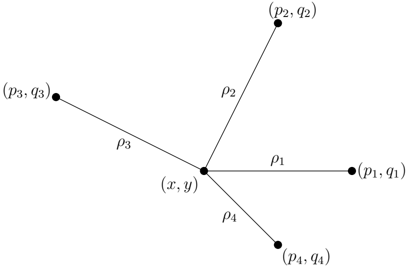

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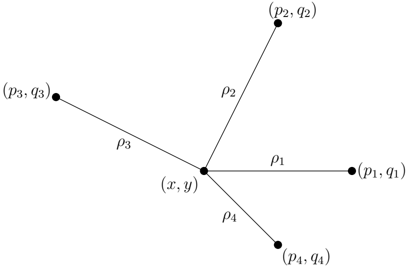

- (x, y) is an unknown location in the plane
- (p_i, q_i) are known locations of beacons for $i = 1, \ldots, n$
- measure distance ρ_i between (x, y) and beacon i

Linearized range measurements

- $\rho \in \mathbb{R}^4$ is a nonlinear function of $(x, y) \in \mathbb{R}^2$:

$$\rho_i(x, y) = \sqrt{(x - p_i)^2 + (y - q_i)^2}$$

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- linearize around (x_0, y_0):

$$\delta\rho \approx A \begin{bmatrix} \delta x \\ \delta y \end{bmatrix},$$

where

$$A_{i1} = \frac{x_0 - p_i}{\sqrt{(x_0 - p_i)^2 + (y_0 - q_i)^2}},$$
$$A_{i2} = \frac{y_0 - q_i}{\sqrt{(x_0 - p_i)^2 + (y_0 - q_i)^2}}$$

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- define matrix product AB as matrix representation of $f \circ g$

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- therefore,

$$(AB)_{ij} = \sum_{k=1}^{p} A_{ik} B_{kj}$$

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- blending measurements

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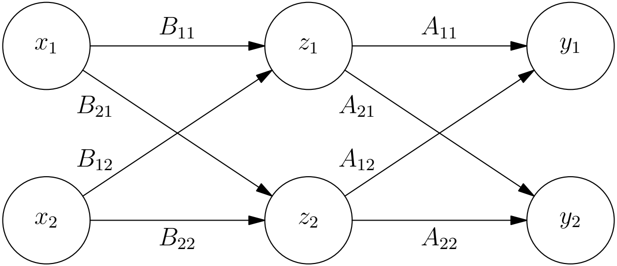
- columns of AB are linear combinations of columns of A
- jth column of B gives coefficients for jth column of AB
- jth column of AB is A times jth column of B
- effects of secondary inputs

Signal flow



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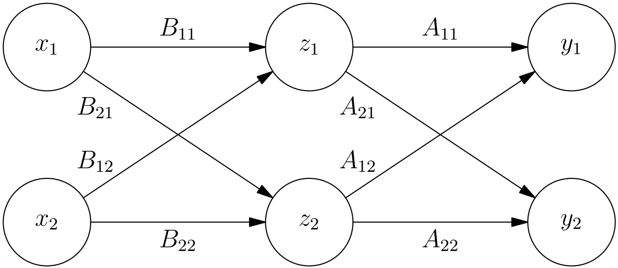

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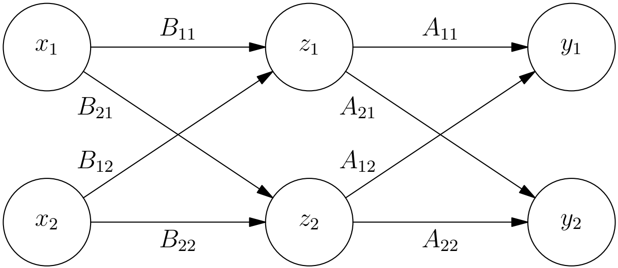

- $A_{ik}B_{kj}$ is gain from input x_i to output y_j through z_k
- $(AB)_{ij} = \sum_{k=1}^{p} A_{ik}B_{kj}$ is total gain from input x_i to output y_j