Linear systems of equations

system of linear equations:

$$\begin{array}{ccccccccc} y_1 & = & A_{11} & x_1 & + & \cdots & + & A_{1n} & x_n \\ & \vdots & & & & & & & \\ y_m & = & A_{m1} & x_1 & + & \cdots & + & A_{mn} & x_n \end{array}$$

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where

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}, \qquad A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

right side of system *defines* matrix-vector multiplication

Linearly constrained least-squares problems

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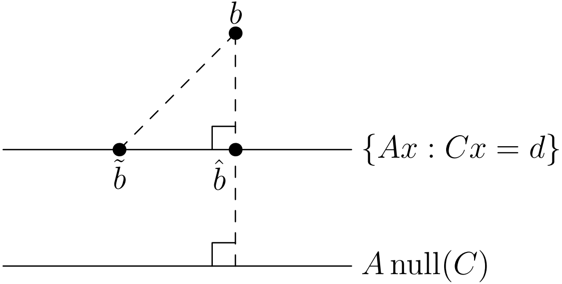
normal equations:

$$\begin{bmatrix} A^\mathsf{T} A & C^\mathsf{T} \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} A^\mathsf{T} b \\ d \end{bmatrix}$$

The orthogonality principle



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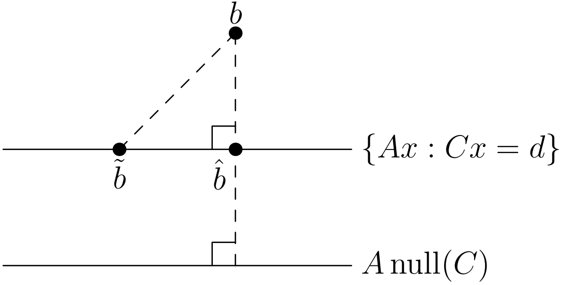

$$b - \hat{b} \perp A\,\mathrm{null}(C) \qquad \Rightarrow \qquad A^{\mathsf{T}}(b - \hat{b}) \in \mathrm{null}(C)^{\perp} = \mathrm{range}(C^{\mathsf{T}})$$

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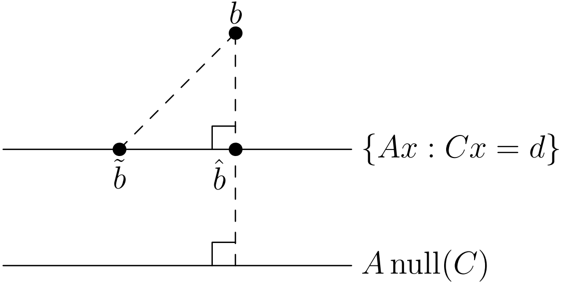

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$$\begin{bmatrix} A^\mathsf{T}A & C^\mathsf{T} \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} A^\mathsf{T}b \\ d \end{bmatrix} \qquad \begin{matrix} \text{orthogonality} \\ \text{feasibility} \end{matrix}$$

Linear dynamical systems

discrete-time linear dynamical system:

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

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solutions of the state and measurement equations:

$$\begin{aligned} x(t) &= A^t x(0) + \sum_{t=0}^{t-1} A^{t-\tau-1} Bu(\tau), \\ y(t) &= CA^t x(0) + \sum_{t=0}^{t-1} CA^{t-\tau-1} Bu(\tau) + Du(t) \end{aligned}$$

Eigenvalues and eigenvectors

suppose $A \in \mathbb{R}^{n \times n}$

- $v \in \mathbb{C}^n$ is (right) eigenvector of A with eigenvalue $\lambda \in \mathbb{C}$ if

$$Av = \lambda v, \qquad v \neq 0$$

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- in matrix form:

$$A \begin{bmatrix} v_1 & \cdots & v_k \end{bmatrix} = \begin{bmatrix} v_1 & \cdots & v_k \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_k \end{bmatrix}$$

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Diagonalizable matrices

- A diagonalizable if there is linearly independent set of n eigenvectors:

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- where

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- if $|\lambda_1| > |\lambda_2| > |\lambda_3|$, then λ_2 is "vice" dominant eigenvalue

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- unique representation using symmetric matrix $A \in \mathbb{S}^n$

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Extremal-trace problems

$$\begin{aligned} \underset{Q \in \mathbb{R}^{n \times k}}{\text{maximize}} &: \operatorname{tr}(Q^{\mathsf{T}} A Q) \\ \text{subject to} &: Q^{\mathsf{T}} Q = I \end{aligned}$$

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- singular-value decomposition:

$$A = U\Sigma V^\mathsf{T}$$

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- $\Sigma \in \mathbb{R}^{r\times r}$ diagonal, nonsingular
- $V \in \mathbb{R}^{n\times r}$ has orthonormal columns

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- image of unit ball under A is ellipsoid with principal axes $\sigma_i u_i$

The singular-value decomposition (dyadic expansion)

$$A = \sum_{i=1}^{r} \sigma_i u_i v_i^{\mathsf{T}}$$

- $\sigma_1 \geq \cdots \geq \sigma_r > 0$ are singular values
- $u_1, \ldots, u_r \in \mathbb{R}^m$ are output singular vectors
- $v_1, \ldots, v_r \in \mathbb{R}^n$ are input singular vectors
- $u_i v_i^{\mathsf{T}}$ is unit atom
  - $\mathrm{rank}(u_i v_i^{\mathsf{T}}) = 1$
  - $\|u_i v_i^{\mathsf{T}}\| = 1$
- SVD decomposes A into sum of unit atoms
  - singular values rank atoms in terms of importance

The pseudoinverse

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- $A^\dagger b$ is least-norm least-squares vector

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} &: \|x\| \\ \text{subject to} &: \|Ax - b\| = \min_z \|Az - b\| \end{aligned}$$

Extremal-trace problems (input)

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- equivalently, solution is matrix Q whose columns are the first k input singular vectors of A

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Low-rank approximation

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best rank-p approximation of A:

$$\hat{A}_p = \sum_{i=1}^{p} \sigma_i u_i v_i^\mathsf{T}$$