EE263: Introduction to Linear Dynamical Systems
Review Session 1

Outline

- administrative information
- examples

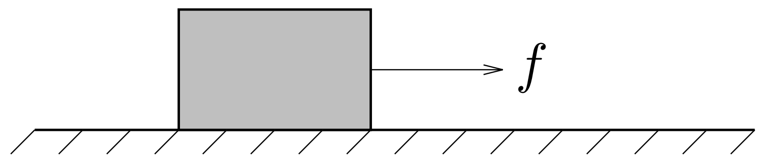
Table as a matrix

- nutrition chart

		Vegetable			
		x_1	x_2	$\cdots$	x_n
Nutrient	y_1	0.50	0.75	$\cdots$	0.9
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	y_m	2.05	0.01	$\cdots$	0.45

- vector $x \in \mathbf{R}^n$ is the vegetable diet; x_j is amount of vegetable j
- vector $y \in \mathbf{R}^m$ is the nutrients; y_i is the amount of nutrient i
- $y = Ax$ gives the nutrients as a function of the vegetable diet
- $A_{ij} =$ amount of nutrient i in 1 unit of vegetable j

Mass/force example



- unit mass, zero position/velocity at $t = 0$, subject to force $f(t)$ for $0 \leq t \leq n$
- $f(t) = x_j$ for $j - 1 \leq t < j$, $j = 1, \ldots, n$
  (x is the sequence of applied forces, constant in each interval)
- let $p(t)$ be the position at time t
- y_1, y_2 are final position and velocity ($i.e.$ $y_1 = p(n)$, $y_2 = \dot{p}(n)$)
- we have $y = Ax$
- find A

Mass/force example (cont...)

solution

- final velocity $y_2 = x_n + \cdots + x_1$
- final position

$$
\begin{aligned}
y_1 &= 0.5(\dot{p}(n) + \dot{p}(n-1)) + \cdots + 0.5(\dot{p}(1) + \dot{p}(0)) \\
&= 0.5\dot{p}(n) + \dot{p}(n-1) + \cdots + \dot{p}(1) \\
&= 0.5(x_n + \cdots + x_1) + (x_{n-1} + \cdots + x_1) + \cdots + x_1 \\
&= 0.5x_n + 1.5x_{n-1} + \cdots + (n-0.5)x_1
\end{aligned}
$$

- write as $y = Ax$

$$
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} n-0.5 & \cdots & 1.5 & 0.5 \\ 1 & \cdots & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}
$$

Linear mechanical system

$$M\ddot{q} + D\dot{q} + Kq = f$$

- $q(t) \in \mathbf{R}^k$ is the vector of deflections
- $f(t) \in \mathbf{R}^k$ is the vector of externally applied forces
- $M, D, K \in \mathbf{R}^{k\times k}$ are the *mass*, *damping*, and *stiffness* matrices, respectively

Linear system equations

- assume $M \in \mathbf{R}^{k \times k}$ full rank
- let state $x = (q, \dot{q}) = [q^T \ \dot{q}^T]$
- let action $u = f$
- let output $y = q$
- write $\dot{x} = Ax + Bu$, $y = Cx + Du$

solution

$$\begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix} \begin{bmatrix} q \\ \dot{q} \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} f$$

$$q = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$

Examples

- $x \in \mathbf{R}^n$
- find A for which $y = Ax$ is the running average of x, *i.e.*,

$$y_i = \frac{1}{i} \sum_{j=1}^{i} x_j, \quad i = 1, \ldots, n$$

Solution.

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1/2 & 1/2 & 0 & 0 & \cdots & 0 \\ 1/3 & 1/3 & 1/3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/n & 1/n & 1/n & 1/n & \cdots & 1/n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Examples (contd...)

- what is $y = Ax$, with $A = (1/n)\mathbf{1}\mathbf{1}^T$?

Solution.

- $\mathbf{1}^T x = x_1 + \cdots + x_n$, the sum of the x_i
- $(1/n)\mathbf{1}^T x$ is the average of the x_i
- $y = (1/n)\mathbf{1}\mathbf{1}^T x$ is a vector of size n, with each component equal to the average of the x_i

Polynomial evaluation

- polynomial $p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_{n-1} t^{n-1}$
- points $t_1, \ldots, t_m \in \mathbf{R}$
- $y_i = p(t_i)$, *i.e.*, p evaluated at points $t_1, \ldots, t_m$
- can write as $y = Ta$:

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} 1 & t_1 & \cdots & t_1^{n-1} \\ 1 & t_2 & \cdots & t_2^{n-1} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & t_m & \cdots & t_m^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

- so y depends *linearly* on a, can do parameter estimation etc.

Node adjacency matrix

- graph with n nodes and (undirected) edges
- node adjacency matrix $A \in \mathbf{R}^{n \times n}$ is given by

$$A_{ij} = \begin{cases} 1 & \text{there is an edge connecting node } i \text{ and node } j \\ 0 & \text{otherwise} \end{cases}$$

- what is meaning of (i, j)th entry of A^k?

Node adjacency matrix (cont...)

Solution.

- let's start with $k = 2$
- $A_{ik}A_{kj} = 1$ only if there is an edge from node i to node k, and an edge from node k to node j, $i.e.$, there is a path of length 2 between i and j, passing through node k
- $(A^2)_{ij} = \sum_{k=1}^{n} A_{ik}A_{kj}$ which is the *number of paths of length* 2 *between* i *and* j
- in general case, $\left(A^k\right)_{ij}$ is the number of paths of length k between nodes i and j

Node incidence matrix

- graph with n nodes and m directed edges

- node incidence matrix $A \in \mathbf{R}^{n \times m}$ is defined as

$$A_{ij} = \begin{cases} 1 & \text{edge } j \text{ enters (points into) node } i \\ -1 & \text{edge } j \text{ leaves (points out of) node } i \\ 0 & \text{otherwise.} \end{cases}$$

- what does $y = Ax$ mean?

- what does $z = A^T w$ mean?

- what is AA^T?

Node incidence matrix (contd...)

Solution.

- defining x_j to be flow rate on edge j, y_i is the total flow into node i (out of, if negative)

- for $w_j \in \{0, 1\}$, z_i is the sum rate on edge i

- $(AA^T)_{ii} =$ number of edges connected to the ith node

- for $i \neq j$, $(AA^T)_{ij} = -1$, only if there is an edge from node i to node j