EE263: Introduction to Linear Dynamical Systems
Review Session 2

Basic concepts from linear algebra

- nullspace
- range
- rank and conservation of dimension

Prerequisites

We assume that you are familiar with the basic definitions of the following concepts from lecture 3:

- vector spaces
- subspaces
- independence
- span
- basis
- dimension

Nullspace of a matrix

- For a matrix $A \in \mathbf{R}^{m \times n}$, the nullspace is defined as,

$$\mathcal{N}(A) = \{\ x \in \mathbf{R}^n \mid Ax = 0\ \}.$$

- Is $\mathcal{N}(A) \subseteq \mathbf{R}^N$ a vector subspace of $\mathbf{R}^n$? Can you prove it?

  Solution: take two vectors $v_1, v_2 \in \mathcal{N}(A)$, and scalars $\alpha_1, \alpha_2 \in \mathbf{R}$. Then,

$$A(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 A v_1 + \alpha_2 A v_2 = 0.$$

  So $\alpha_1 v_1 + \alpha_2 v_2 \in \mathcal{N}(A)$.

- Roughly speaking, to verify that a set $\mathcal{V} \subseteq \mathbf{R}^n$ is a subspace, we need only check that it is closed under vector addition and scalar multiplication.

Example 1

Let

$$P = \begin{bmatrix} A \\ A+B \\ A+B+C \end{bmatrix}.$$

- True or false?

$$\mathcal{N}(P) = \mathcal{N}(A) \cap \mathcal{N}(B) \cap \mathcal{N}(C)$$

- Note that $\mathcal{N}(P)$ is a set, and $\mathcal{N}(A) \cap \mathcal{N}(B) \cap \mathcal{N}(C)$ is also a set.

- We say that two sets X and Y are equal if $z \in X \Rightarrow z \in Y$, and $z \in Y \Rightarrow z \in X$.

  Solution:

- syntax check: Two subspaces can only be equal if they contain vectors of the same size:

LHS: if $A \in \mathbf{R}^{m\times n}$, then $B \in \mathbf{R}^{m\times n}$ and $C \in \mathbf{R}^{m\times n}$. Therefore, P has n columns, and so $\mathcal{N}(P)$ is a subspace of $\mathbf{R}^n$.

RHS: $\mathcal{N}(A)$, $\mathcal{N}(B)$, and $\mathcal{N}(C)$ are all subspaces of $\mathbf{R}^n$, and hence their intersection must also be a subspace of $\mathbf{R}^n$.

- show that $x \in \mathcal{N}(P) \Rightarrow \mathcal{N}(A) \cap \mathcal{N}(B) \cap \mathcal{N}(C)$.

  Let $x \in \mathcal{N}(P)$. This implies

$$\left.\begin{array}{c} Ax = 0 \\ Ax + Bx = 0 \\ Ax + Bx + Cx = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{c} x \in \mathcal{N}(A) \\ x \in \mathcal{N}(B) \\ x \in \mathcal{N}(B) \end{array}\right\} x \in \mathcal{N}(A) \cap \mathcal{N}(B) \cap \mathcal{N}(C).$$

- show that $x \in \mathcal{N}(A) \cap \mathcal{N}(B) \cap \mathcal{N}(C) \Rightarrow x \in \mathcal{N}(P)$

  This is trivial: if $Ax = 0$, $Bx = 0$, $Cx = 0$ then $Ax + Bx = 0$ and $Ax + Bx + Cx = 0$, so $x \in \mathcal{N}(P)$.

Example 2

Is this true or false?

$$\mathcal{N}(A^T A) = \mathcal{N}(A).$$

Solution:

- syntax check: If $A \in \mathbf{R}^{m\times n}$ then $A^T A \in \mathbf{R}^{n\times n}$, so $\mathcal{N}(A^T A)$ and $\mathcal{N}(A)$ are both subspaces of $\mathbf{R}^n$.

- show that $x \in \mathcal{N}(A) \Rightarrow x \in \mathcal{N}(A^T A)$:

$$Ax = 0 \Rightarrow (A^T A)x = A^T(Ax) = A^T 0 = 0,$$

so $x \in \mathcal{N}(A^T A)$.

- show that $x \in \mathcal{N}(A^T A) \Rightarrow x \in \mathcal{N}(A)$:

  Suppose $x \in \mathcal{N}(A^T A)$. Then, $A^T A x = 0$ and so

$$x^T A^T A x = (Ax)^T (Ax) = \|Ax\|^2 = 0 \Rightarrow \|Ax\| = 0.$$

  - norm of a vector is zero if and only if the vector is equal to zero, $i.e.$, $\|z\| = 0 \Leftrightarrow z = 0$.
  - $\|Ax\| = 0$ therefore implies that $Ax = 0$, and so $x \in \mathcal{N}(A)$.

Range of a Matrix

- The range of a matrix $A \in \mathbf{R}^{m \times n}$ is defined as,

$$\mathcal{R}(A) = \{ \ Ax \mid x \in \mathbf{R}^n \ \}$$

- $\mathcal{R}(A)$ is a subspace of $\mathbf{R}^m$ (check this for yourself!).

- Roughly speaking, it is the set of vectors in $\mathbf{R}^m$ that can be 'hit' by the linear mapping $y = Ax$.

- Given a set of linear equations $y = Ax$ (where y and A are known), a solution x exists if and only if y can be hit by the linear mapping $y = Ax$, $i.e.$, $y \in \mathcal{R}(A)$.

Two more useful subspaces

- $\mathcal{R}(A^T)$: the space spanned by the rows of A^T; also called *the row-space*.
- $\mathcal{N}(A^T)$: also called the *left nullspace*.
- $\mathcal{R}(A)$, $\mathcal{N}(A)$, $\mathcal{R}(A^T)$, $\mathcal{N}(A^T)$ are sometimes called the four fundamental subspaces,
- By the end of lecture 4, you will understand the relationships between these subspaces and their properties.

Example 3

Draw $\mathcal{R}(A)$, $\mathcal{N}(A)$, $\mathcal{R}(A^T)$, $\mathcal{N}(A^T)$ for the matrix,

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}. \tag{1}$$

Solution:

- $\mathcal{R}(A)$: note that A has only one independent column: $\begin{bmatrix} 1 & 0 \end{bmatrix}^T$.
  - So $\mathcal{R}(A)$ is all the scalar multiples of $\begin{bmatrix} 1 & 0 \end{bmatrix}^T$.
- $\mathcal{R}(A^T)$: A^T also has only one independent column, $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$.
  - So $\mathcal{R}(A^T)$ is all the scalar multiples of $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$.
- Using a similar argument for $\mathcal{N}(A)$ and $\mathcal{N}(A^T)$ gives us the diagrams shown in figure 1. (Do this yourself!)

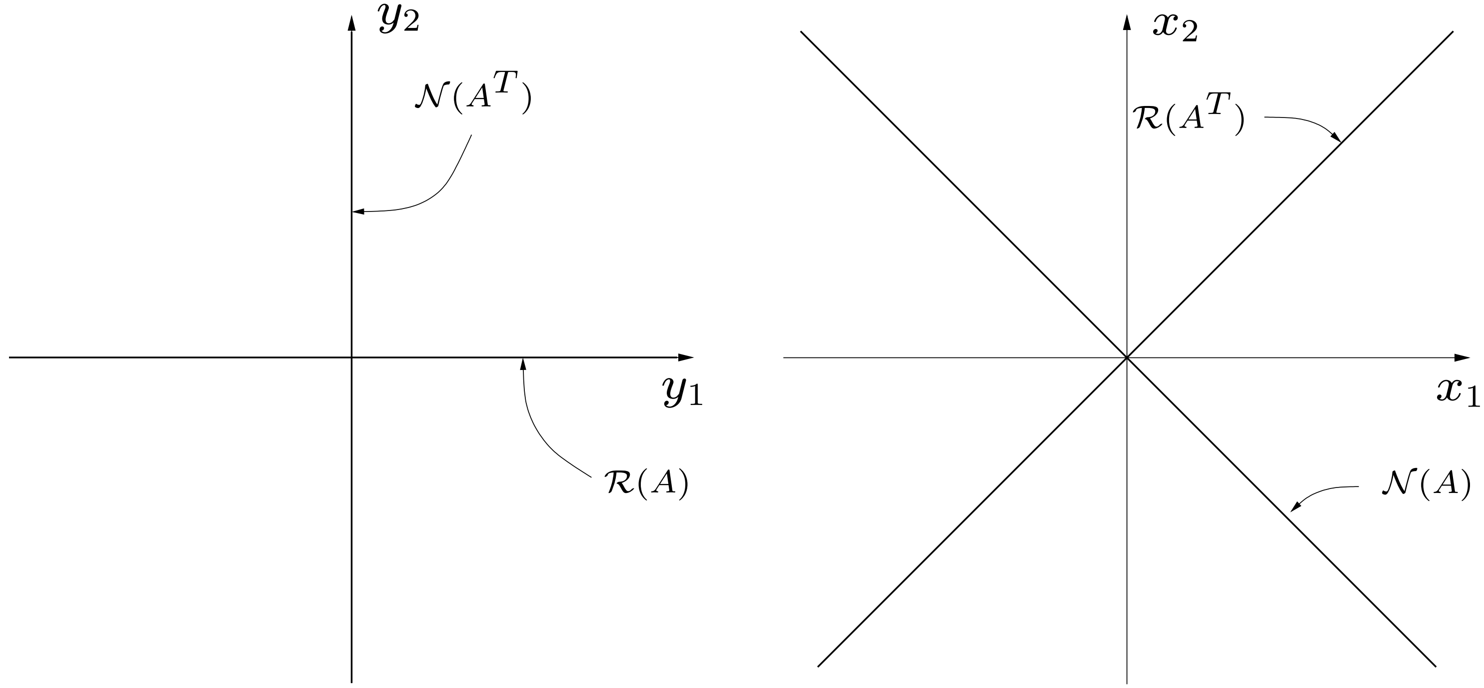

Figure 1: Four subspaces for the matrix in (1).

- Vectors in $\mathcal{R}(A)$ are orthogonal to vectors in $\mathcal{N}(A^T)$; vectors in $\mathcal{R}(A^T)$ are orthogonal to vectors in $\mathcal{N}(A)$.
- This is no coincidence, and in fact, in lecture 4, you'll see why this is true.

Rank and conservation of dimension

The rank of a matrix $A \in \mathbf{R}^{m\times n}$ defined as

$$\mathbf{rank}(A) = \dim \mathcal{R}(A).$$

Some facts:

- $\mathbf{rank}(A) = \mathbf{rank}(A^T)$.
- $\mathbf{rank}(A)$ is the maximum number of independent columns (or rows) of A, hence $\mathbf{rank}(A) \le \min(m, n)$.
- $\mathbf{rank}(A) + \dim \mathcal{N}(A) = n$. (Conservation of dimension)
  - $\mathbf{rank}(A)$ is the dimension of the set 'hit' by the mapping $y = Ax$,
  - $\mathcal{N}(A)$ is the dimension of the set of x mapped to zero by $y = Ax$.
  - n is the number of degrees of freedom in x.
  - so, roughly speaking, each degree of freedom in the input is either mapped to zero, or ends up in the output.

Example 4

- What is $\mathbf{rank}(A)$, the matrix labeled (1), from the previous example? *answer:* 1.

- Given $A \in \mathbf{R}^{m\times n}$, what is
  - $\dim \mathcal{R}(A^T) + \dim \mathcal{N}(A)$? *answer*: n.
  - $\dim \mathcal{R}(A) + \dim \mathcal{N}(A^T)$? *answer*: m.

Full rank matrices

$A \in \mathbf{R}^{m \times n}$ is *full rank* if $\mathbf{rank}(A) = \min(m, n)$.

- For square matrices, full rank means the matrix is nonsingular and invertible.
- For skinny matrices ($m \geq n$), full rank means that the columns of A are independent, and sometimes we say that A is *full column rank*.
- For fat matrices ($m \leq n$), full rank means that the rows of A are independent, sometimes referred to as *full row rank*.

Example 5

Suppose $A \in \mathbf{R}^{m\times n}$ is fat ($m < n$) and full rank. Show that A has a *right inverse*, *i.e.*, there is a matrix $B \in \mathbf{R}^{n\times m}$ such that $AB = I$. Is the right inverse unique?

Solution:

- If A is fat and full rank, then $\mathbf{rank}(A) = m$, which means that $\dim \mathcal{R}(A) = m$, and so $\mathcal{R}(A) = \mathbf{R}^m$ (*i.e.*, A is an *onto* matrix).
- Therefore, the equation $Ax = y$ can be solved for any $y \in \mathbf{R}^m$.
- So we can find $[b_1, \ldots, b_m]$ that satisfy

$$Ab_i = e_i, \quad i = 1, \ldots, m.$$

- Form $B = [b_1, \ldots, b_m]$. Then,

$$AB = A\left[\begin{array}{cccc} b_1 & b_2 & \cdots & b_m \end{array}\right] = \left[\begin{array}{cccc} e_1 & e_2 & \cdots & e_m \end{array}\right] = I,$$

so B is a right inverse of A.

- uniqueness:
  - By the conservation of dimension

$$\dim \mathcal{N}(A) = n - \dim \mathcal{R}(A) = n - m \geq 1.$$

  - Let's take a nonzero vector $z \in \mathcal{N}(A)$, and any nonzero vector $w \in \mathbf{R}^m$, then for any right inverse B,

$$A(B + zw^T) = AB + Azw^T = I + 0 = I,$$

    so $B + zw^T$ is also a right inverse.
  - Since B and $B + zw^T$ are different, we have two different right inverses. So the right inverse is not unique.

Example 6

Suppose $A \in \mathbf{R}^{m \times n}$ is skinny ($m > n$) and full rank. Show that A has a *left inverse*, *i.e.*, there is a matrix $B \in \mathbf{R}^{n \times m}$ such that $BA = I$.

Solution.

- If A is skinny and full rank, then $\mathbf{rank}(A) = n$, which means that A has n independent rows and columns.

- Thus A^T is fat and full rank, and so the equation $A^T x = y$ can be solved for any $y \in \mathbf{R}^n$.

- Similar to the previous example, we can find a matrix $C \in \mathbf{R}^{m \times n}$ such that $A^T C = I$.

- Let $B = C^T$. B is a left inverse of A.