EE263: Introduction to Linear Dynamical Systems
Review Session 5

Outline

- eigenvalues and eigenvectors
- diagonalization
- matrix exponential

Eigenvalues and eigenvectors

- we say that $\lambda \in \mathbf{C}$ is an eigenvalue of a square matrix $A \in \mathbf{C}^{n\times n}$ if

$$\mathcal{X}(\lambda) = \det(\lambda I - A) = 0$$

  where $\mathcal{X}(s) = \det(sI - A)$ is the *characteristic polynomial* of the matrix A

- $\det(\lambda I - A) = 0$ implies $\lambda I - A$ is singular, so $\exists$ a nonzero vector $v \in \mathcal{N}(\lambda I - A) \subseteq \mathbf{C}^n$, *i.e.*

$$Av = \lambda v$$

- we say that v is an eigenvector of A (associated with the eigenvalue λ)
- eigenvectors must be nonzero by definition
- the concept of eigenvalues and eigenvectors applies only to square matrices

Basic facts

let $A \in \mathbf{C}^{n\times n}$, how many eigenvalues does A have?

- an eigenvalue is a root of the polynomial $\mathcal{X}(s)$, which is of degree n
- $\mathcal{X}(s)$ must have exactly n roots (counting multiplicities), and so A must have exactly n eigenvalues (but these need not be distinct)
- we can write the polynomial $\mathcal{X}(s)$ as

$$\mathcal{X}(s) = (s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)$$

- which implies that

$$\det(-A) = \mathcal{X}(0) = \Pi_{i=1}^{n}(-\lambda_i)$$

or more simply

$$\det(A) = \Pi_{i=1}^{n}\lambda_i$$

Basic facts contd.

is the eigenvector associated with an eigenvalue unique?

- if v is an eigenvector of A, then $Av = \lambda v$
- this means that $\alpha A v = \alpha \lambda v$ for any $\alpha \in \mathbf{C}$, $\alpha \neq 0$, and so αv is also an eigenvector

from now on we shall assume that A is real

- even when A is real, the eigenvalues and eigenvectors of A can be complex
- when both A and λ are real, we can always find a real eigenvector v
- if A is real and λ is complex, then $\overline{\lambda}$ is also an eigenvalue of A, and $\overline{v}$ is an eigenvector associated with $\overline{\lambda}$

Idempotent matrices

suppose $A \in \mathbf{R}^{n \times n}$, $A^2 = A$ (A is an *idempotent* matrix), what can we say about the eigenvalues of A?

- let's suppose λ is an eigenvalue of A, then $Av = \lambda v$ for some nonzero vector v
- since $A^2 = A$, we can write

$$\lambda v = Av = A^2 v = AAv = \lambda^2 v$$

- by definition, v must be a nonzero vector, so we must have $\lambda = \lambda^2$, which implies that $\lambda(\lambda - 1) = 0$
- the eigenvalues of A must be either 0 or 1
- cf. projection matrices

Left eigenvectors

given $A \in \mathbf{R}^{n \times n}$, show that the eigenvalues of A and A^T are the same

- λ is an eigenvalue of A if $\det(\lambda I - A) = 0$, we know that for any square matrix B, $\det(B) = \det(B^T)$, and so

$$0 = \det(\lambda I - A) = \det((\lambda I - A)^T) = \det(\lambda I - A^T).$$

- $i.e.$ any eigenvalue of A must be an eigenvalue of A^T

what is the relationship between the eigenvectors of A and A^T?

- let's suppose w is an eigenvector of A^T associated with the eigenvalue λ, so $A^T w = \lambda w$. We can write this as

$$w^T A = \lambda w^T,$$

- $i.e.$ w is a left eigenvector of A, associated with the eigenvalue λ

Example

find the eigenvalues of the following matrix:

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 3 & 9 \\ 0 & 0 & 5 \end{bmatrix}$$

- the characteristic polynomial of A is

$$\det(sI - A) = \det\left(\begin{bmatrix} s-1 & -3 & -2 \\ 0 & s-3 & -9 \\ 0 & 0 & s-5 \end{bmatrix}\right) = (s-1)(s-3)(s-5).$$

  (note: the determinant of a triangular matrix is the product of the entries along the diagonal)

- the eigenvalues of A are the roots of the characteristic polynomial, *i.e.*, 1, 3, and 5

Finding eigenvalues

- can determine eigenvalues and eigenvectors of matrices analytically only in very special cases: for example, when the matrix is triangular, or idempotent, or when the matrix is in $\mathbf{R}^{2\times 2}$ or $\mathbf{R}^{3\times 3}$ (in which case we can easily find the roots of the characteristic polynomial)

- in general, we need to use numerical algorithms to find eigenvalues and eigenvectors

- in matlab we can use the function `eig`:

```
[U,V]=eig(A)
```

- puts normalized eigenvectors as columns in U and the associated eigenvalues as the diagonal entries in V

Eigenvalues of matrix products

for any $A \in \mathbf{R}^{m\times n}$, $B \in \mathbf{R}^{n\times m}$, Sylvester's determinant identity states that

$$\det(I_m - AB) = \det(I_n - BA),$$

using this, show that the nonzero eigenvalues of AB and BA are identical

- suppose $\lambda \neq 0$, then

$$\begin{aligned}\det(\lambda I_n - BA) &= \lambda^n \det(I_n - B(\lambda^{-1}A)) \\ &= \lambda^n \det(I_m - (\lambda^{-1}A)B) \\ &= \lambda^{n-m} \det(\lambda I_m - AB)\end{aligned}$$

- if λ is a nonzero eigenvalue of BA then $\det(\lambda I_n - BA) = 0$
- implies that $\lambda^{n-m} \det(\lambda I_m - AB) = 0$.
- must have $\det(\lambda I_m - AB) = 0$, so λ is also an eigenvalue of AB

Rank one matrices

suppose $A \in \mathbf{R}^{n\times n}$ is a rank 1 matrix $\big(i.e.$, $\mathbf{rank}(A) = 1\big)$, then A can be expressed as $A = xy^T$, for nonzero vectors $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^n$

- suppose $y^T x \neq 0$, find a nonzero eigenvalue of A, and a left and a right eigenvector associated with this nonzero eigenvalue
- what are the other eigenvalues of A?

solution

- let's consider the product Ax; we can write

$$Ax = (xy^T)x = (y^T x)x.$$

this shows that $y^T x$ is an eigenvalue of A, and x is an associated eigenvector

- similarly,
$$y^T A = y^T(xy^T) = (y^T x)y^T$$
so y is a left eigenvector corresponding to the eigenvalue $y^T x$.

- by the previous example, the nonzero eigenvalues of xy^T must be the same as the nonzero eigenvalues of $y^T x$. But $y^T x$ is just a scalar, with only one eigenvalue $y^T x$, and so A can only have one nonzero eigenvalue, hence all the other eigenvalues of A are zero

Diagonalization

we say that a matrix $A \in \mathbf{R}^{n \times n}$ is *diagonalizable* if A has a set of linearly independent eigenvectors $v_1, \ldots, v_n$, in this case

$$A \left[\begin{array}{ccc} v_1 & \cdots & v_n \end{array} \right] = \left[\begin{array}{ccc} v_1 & \cdots & v_n \end{array} \right] \left[\begin{array}{ccc} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{array} \right]$$

defining $T = \left[\begin{array}{ccc} v_1 & \cdots & v_n \end{array} \right]$, and $\Lambda = \mathbf{diag}(\lambda_1, \ldots, \lambda_n)$, we can write

$$AT = T\Lambda$$

and so

$$T^{-1}AT = \Lambda$$

conversely, if we can find a nonsingular T, such that $T^{-1}AT = \Lambda$ where Λ is a diagonal matrix, then $AT = T\Lambda$, and so the diagonal entries of Λ are the eigenvalues of A, and the columns of T are an independent set of eigenvectors of A, so an alternative definition is that a matrix is diagonalizable if there exists a nonsingular T, such that $T^{-1}AT$ is a diagonal matrix

importantly:

- if A has distinct eigenvalues, $i.e.$, $\lambda_i \neq \lambda_j$ for $i \neq j$, then A is diagonalizable.

the converse is false, $e.g.$ the identity matrix I is diagonalizable (indeed, it is already a diagonal matrix), but all the eigenvalues are equal to 1

we'll see that in many cases diagonalization can simplify matrix expressions, and offer insight into the properties of matrices

Example

let A be a rank-1 matrix $\Leftrightarrow$ A can be expressed as $A = xy^T$ for nonzero vectors x and y; suppose $y^T x \neq 0$, show that A is diagonalizable

solution

- we know $A = xy^T$ has a nonzero eigenvalue $y^T x$, and x is a corresponding eigenvector
- also $y^T x$ is the only nonzero eigenvalue, the other $n-1$ eigenvalues are all zero
- if v is an eigenvector corresponding to the eigenvalue 0, then

$$v \in \mathcal{N}(0I - A) = \mathcal{N}(A)$$

- but since $\mathbf{rank}(A) = 1$, we know that $\dim(\mathcal{N}(A)) = n - 1$, and so we can pick $n-1$ independent eigenvectors $v_2, \ldots, v_n \in \mathcal{N}(A)$, that correspond to the eigenvalue 0.

- by construction $\{v_2, \dots, v_n\}$ is independent, so we need to show that x is independent of $v_2, \dots, v_n$, *i.e.*, x cannot be expressed as a linear combination of $v_2, \dots, v_n$
- suppose that we can find $\alpha_2, \dots, \alpha_n$, such that

  $$x = \alpha_2 v_2 + \cdots + \alpha_n v_n.$$

  then

  $$Ax = \alpha_2 A v_2 + \cdots + \alpha_n A v_n = 0.$$

  but we know that $Ax = (y^T x)x \neq 0$, which leads to a contradiction
- so x cannot depend on the vectors $v_2, \dots, v_n$, thus, the set of eigenvectors $\{x, v_2, \dots, v_n\}$ is independent, which shows that A is diagonalizable

Example

suppose $A \in \mathbf{R}^{n\times n}$ is diagonalizable, how can we find a matrix B such that $B^2 = A$? (B is a square root of A)

if A is diagonalizable, then $A = T\Lambda T^{-1}$, where $\Lambda = \mathbf{diag}(\lambda_1, \ldots, \lambda_n)$, let $\mu_1^2 = \lambda_1, \ldots, \mu_n^2 = \lambda_n$, and let $B = T\,\mathbf{diag}(\mu_1, \ldots, \mu_n)T^{-1}$

$$\begin{aligned}
B^2 &= T\,\mathbf{diag}(\mu_1, \ldots, \mu_n)T^{-1}T\,\mathbf{diag}(\mu_1, \ldots, \mu_n)T^{-1} \\
&= T\,\mathbf{diag}(\mu_1, \ldots, \mu_n)\,\mathbf{diag}(\mu_1, \ldots, \mu_n)T^{-1} \\
&= T\,\mathbf{diag}(\mu_1^2, \ldots, \mu_n^2)T^{-1} \\
&= T\Lambda T^{-1} \\
&= A
\end{aligned}$$

a matrix square root is not unique, using this method we can choose 2^n different square roots (depending on whether we choose $\mu_i = +\sqrt{\lambda_i}$, or $\mu_i = -\sqrt{\lambda_i}$)), however, not all matrix square roots are given this way for example, the matrix

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

is diagonalizable, and if we use the method we just described we find that the square root is simply the zero matrix

an alternative square root is

$$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

if you want to compute the square root of a matrix in matlab you can use the function `sqrtm`

Simultaneously diagonalizable matrices

two matrices $A \in \mathbf{R}^{n\times n}$ and $B \in \mathbf{R}^{n\times n}$ are simultaneously diagonalizable if we can find a matrix T, such that $T^{-1}AT = \Lambda_A$, and $T^{-1}BT = \Lambda_B$, where Λ_A, and Λ_B are diagonal

show that if A and B are simultaneously diagonalizable, then $AB = BA$

solution

- write $A = T\Lambda_A T^{-1}$, and $B = T\Lambda_B T^{-1}$
- so

$$\begin{aligned} AB &= T\Lambda_A T^{-1} T\Lambda_B T^{-1} \\ &= T\Lambda_A \Lambda_B T^{-1} \\ &= T\Lambda_B \Lambda_A T^{-1} \\ &= T\Lambda_B T^{-1} T\Lambda_A T^{-1} \\ &= BA, \end{aligned}$$

since diagonal matrices commute under matrix multiplication

- with a little more work we can show that two diagonalizable matrices A and B commute (under matrix multiplication), if and only if they are simultaneously diagonalizable

Linear dynamical systems and invariant sets

a continuous time linear dynamical system has the form

$$\dot{x} = Ax$$

the solution of the linear dynamical system is

$$x(t) = e^{tA}x(0)$$

where

$$e^{A} = I + A + \frac{A^2}{2!} + \cdots$$

is the matrix exponential of A

we say that a set $S \in \mathbf{R}^n$ is invariant under $\dot{x} = Ax$ if $x(t) \in S$ implies $x(\tau) \in S$ for all $\tau \geq t$, this means that whenever the trajectory enters the set S, it must stay in S

Example

consider a linear dynamical system $\dot{x} = Ax$, where $x(t) \in \mathbf{R}^n$

the positive quadrant $\mathbf{R}^n_+$ is the set of vectors whose components are all nonnegative, *i.e.*, $\mathbf{R}^n_+ = \{x \mid x_i \geq 0,\ i = 1, \ldots, n\}$

we say the system is positive quadrant invariant (PQI) if whenever we have $x(T) \in \mathbf{R}^n_+$, we have $x(t) \in \mathbf{R}^n_+$, for all $t \geq T$

what are the conditions on A for which the system $\dot{x} = Ax$ is PQI?

- we notice that at the quadrant boundaries, the derivative $\dot{x} = Ax$ must point into the quadrant
- so whenever $x_i = 0$, and $x_j \geq 0$, $j \neq i$, we must have $e_i^T Ax \geq 0$
- this is equivalent to the condition

$$A_{ij} \geq 0, \quad j = 1, \ldots, i-1, i+1, \ldots, n$$

in other words, the off-diagonal elements must be non-negative