# EE263: Introduction to Linear Dynamical Systems
# Review Session 6

Outline

- diagonalizability
- eigen decomposition theorem
- applications (modal forms, asymptotic growth rate)

# Diagonalizability

- consider square matrix $A \in \mathbf{R}^{n \times n}$. Assume that $\lambda \in \mathbf{C}$ is an eigenvalue of A
- then, $A$ is diagonalizable if and only if the multiplicity of $\lambda_i$ equals $\dim(\mathcal{N}(\lambda_i I - A)) = n - \mathbf{rank}(\lambda_i I - A)$, for all $i$

# Diagonalizability

- to see this, note that we can write the characteristic polynomial of $A$ as

$$\chi(\lambda) = (\lambda - \lambda_1)^{n_1}(\lambda - \lambda_2)^{n_2} \ldots (\lambda - \lambda_k)^{n_k}$$

- the eigenvectors corresponding to $\lambda_i$ are given by the linear system

$$AX = \lambda_i X$$

  or $(A - \lambda_i I)X = 0$

- so if $\dim \mathcal{N}(\lambda_i I - A) = n_i$, then we can find $n_i$ independent eigenvectors corresponding to $\lambda_i$

- if this condition holds for all $i$, then the matrix $A$ is diagonalizable (and vice versa)

# Example

is this matrix diagonalizable?

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \in \mathbf{R}^{3\times 3}$$

*Solution.*

- let's write the characteristic polynomial for $A$: $\chi(A) = \det(\lambda I - A) = (\lambda - 4)(\lambda - 3)^2$
- eigenvalues are $\lambda_1 = 4$ with multiplicity 1, and $\lambda_2 = 3$ with multiplicity 2
- the condition is satisfied for $\lambda_1$, since $\lambda_1$ has multiplicity 1

- for $\lambda_2$, since
$$\lambda_2 I - A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$
has rank 2, we have $n - \mathbf{rank}(\lambda_2 I - A) = 3 - 2 = 1$, which is not the multiplicity of $\lambda_2$

- this means that we cannot find two independent eigenvectors corresponding to $\lambda_2$

- thus $A$ is not diagonalizable (in fact, $A$ is in Jordan canonical form)

# Example

show that the following statement is true: if $A \in \mathcal{C}^{n \times n}$ has distinct eigenvalues, $i.e.$, $\lambda_i \neq \lambda_j$ for $i \neq j$, then $A$ is diagonalizable (Lecture 11-22).

*Solution.* the characteristic polynomial of $A$ has order of $n$. since its roots are distinct, we have $n$ different eigenvalues with multiplicity of $1$.

without loss of generality, let's show $\dim(\mathcal{N}(\lambda_1 I - A)) = 1$

- clearly, $v_i \neq \alpha v_1$ for $i = 2, \ldots, n$
- since $v_1 \notin \mathbf{span}\{v_2, \ldots, v_n\}$, $\{v_1, \ldots, v_n\}$ is linearly independent
- also $v_i \notin \mathcal{N}(\lambda_1 I - A)$ for $i = 2, \ldots, n$ which implies $\dim(\mathcal{N}(\lambda_1 I - A)) = 1$
- therefore, $\dim(\mathcal{N}(\lambda_i I - A)) = 1$ for $i = 1, \ldots, n$

# Example

show that the following statement is true: It is possible for $A$ to have repeated eigenvalues, but still be diagonalizable (Lecture 11-22)

*Solution.*

- as an example, assume

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

  with $\mathcal{X}(\lambda) = (\lambda - 1)^2(\lambda - 2)$ has multiplicity of $2$ for $\lambda_1 = 1$ but still is diagonalizable

- here, $e_1$ and $e_2$ are eigenvectors for $\lambda_1$, and $e_3$ is for $\lambda_2$, so the condition still holds

# Eigen decomposition theorem

- if a matrix $A \in \mathbf{C}^{n \times n}$ is diagonalizable, $A$ can be written as an eigen decomposition

$$\begin{aligned} A &= T \Lambda T^{-1} \\ &= \sum_{i=1}^{n} \lambda_i v_i w_i^T \end{aligned}$$

  where $T = [v_1 \ \cdots \ v_n]$, $\Lambda = \mathbf{diag}(\lambda_1, \ldots, \lambda_n)$, and $v_i$ is the eigenvector corresponding to eigenvalue $\lambda_i$

- if $A$ is symmetric, *i.e.*, $A = A^T$, then $A$ is diagonalizable and can be written as

$$\begin{aligned} A &= Q \Lambda Q^T \\ &= \sum_{i=1}^{n} \lambda_i q_i q_i^T \end{aligned}$$

where $Q = [q_1 \ \cdots \ q_n]$ is an orthogonal matrix

- you may see that this is useful for positive definiteness soon!

# Application: Modal form

suppose $A \in \mathbf{C}^{n\times n}$ is diagonalizable, $i.e.$, $A = T\Lambda T^{-1}$

then we can rewrite $\dot{x} = Ax$ as $\dot{\tilde{x}} = \Lambda \tilde{x}$ where $\tilde{x} = T^{-1}x$

# Application: Real modal form

when eigenvalues (hence $T$) are complex, system can be put in *real modal form*

- assume $A \in \mathbf{R}^{n\times n}$ is diagonalizable, $i.e.$, $A = T\Lambda T^{-1}$, and has a complex eigenvalue $\lambda_i = \sigma_i + j\omega_i$ with corresponding eigenvector $v_i$, $i.e.$, $Av_i = \lambda_i v_i$
- taking the conjugate, we have $A\bar{v}_i = \bar{\lambda}_i\bar{v}_i$, which implies $\bar{v}_i$ is also an eigenvector corresponding to the eigenvalue $\bar{\lambda}_i$
- we can select $\lambda_{i+1} = \bar{\lambda}_i = \sigma_i - j\omega_i$

from this argument, we can show:

$$\begin{aligned} Av_i &= \lambda_i v_i \\ &= (\sigma_i + j\omega_i)(\Re(v_i) + j\Im(v_i)) \end{aligned}$$

$$= \quad \sigma_i \Re(v_i) - \omega_i \Im(v_i) + j(\omega_i \Re(v_i) + \sigma_i \Im(v_i))$$

and

$$\begin{aligned} Av_{i+1} &= A\bar{v}_i \\ &= \bar{\lambda}_i \bar{v}_i \\ &= (\sigma_i - j\omega_i)(\Re(v_i) - j\Im(v_i)) \\ &= \sigma_i \Re(v_i) - \omega_i \Im(v_i) - j(\omega_i \Re(v_i) + \sigma_i \Im(v_i)) \end{aligned}$$

and thus,

$$\begin{aligned} A\Re(v_i) &= \sigma_i \Re(v_i) - \omega_i \Im(v_i) \\ A\Im(v_i) &= \omega_i \Re(v_i) + \sigma_i \Im(v_i) \end{aligned}$$

which can be rewritten in a matrix form:

$$[\Re(v_i)\ \Im(v_i)]\left[\begin{array}{cc} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{array}\right] = A\,[\Re(v_i)\ \Im(v_i)]$$

now, let's bundle the real and complex eigenvalues together. assume $\lambda_i \in \mathbf{R}$ for $i = 1, \dots, r$ and $\lambda_i \in \mathbf{C} \setminus \mathbf{R}$ for $i = r+1, \dots, n$ and let

$$\Lambda_r = [\lambda_1\ \cdots\ \lambda_r], \quad M_i = \left[\begin{array}{cc} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{array}\right],$$

and

$$S = \left[\begin{array}{cccccccc} v_1 & \cdots & v_r & \Re(v_{r+1}) & \Im(v_{r+1}) & \Re(v_{r+3}) & \Im(v_{r+3}) & \cdots \end{array}\right]$$

since $S$ is invertible, we finally have

$$S^{-1}AS = \mathbf{diag}(\Lambda_r, M_{r+1}, M_{r+3}, \dots, M_{n-1})$$

# Example: Standard form for LDS

- given LDS as $\dot{x} = Ax + Bu$ and $y = Cx$
- suppose $A$ is diagonalizable, $i.e.$, $A = T\Lambda T^{-1}$
- let's change coordinates with $x = T\tilde{x}$

then,

$$\begin{aligned}
\dot{\tilde{x}} &= T^{-1}\dot{x} \\
&= T^{-1}AT\tilde{x} + T^{-1}Bu \\
&= \Lambda\tilde{x} + \tilde{B}u \\
y &= CT\tilde{x} \\
&= \tilde{C}\tilde{x}
\end{aligned}$$

where $\tilde{B} = T^{-1}B$ and $\tilde{C} = CT$

# Example

- suppose that $A \in \mathbf{R}_{+}^{n\times n}$ is diagonalizable, and its eigenvalues are nonnegative.

- consider a discrete-time linear dynamic system, $x(t+1) = Ax(t)$, and a given nonnegative initial state, $i.e.$, $x_i(0) \geq 0$ for $i = 1, \ldots, n$

- how can we find the asymptotic growth rate of sum, $i.e.$, $\frac{\mathbf{1}^T x(t+1)}{\mathbf{1}^T x(t)}$ as $t \to \infty$?

*Solution.*

- since $A$ is diagonalizable, we can rewrite $A = \sum_{i=1}^{n} \lambda_i v_i w_i^T$ as above

- without loss of generality let's assume that $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$, and let's take $m$ as the index of the largest eigenvalue for which $x(0$ is not in the associated nullspace, *i.e.*, $w_m^T x(0) \neq 0$, and $w_i^T x(0) = 0$ for $i = 1, ..., m-1$

- then, $\mathbf{1}^T x(t) = \mathbf{1}^T A^t x(0) = \sum_{i=1}^n \lambda_i^t (\mathbf{1}^T v_i)(w_i^T x(0))$

- as $t \to \infty$, we get $\mathbf{1}^T x(t) \approx \lambda_m^t (\mathbf{1}^T v_m)(w_m^T x(0))$

- therefore, $\frac{\mathbf{1}^T x(t+1)}{\mathbf{1}^T x(t)} \to \lambda_m$ as $t \to \infty$

# Transfer matrix and impulse matrix

- assume LDS is given as $\dot{x}(t) = Ax(t) + Bu(t)$ and $y(t) = Cx(t) + Du(t)$
- taking Laplace transform, we can get:

$$\begin{aligned} & sX(s) - x(0) = AX(s) + BU(s) \\ \Leftrightarrow \quad & X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s) \\ \Rightarrow \quad & Y(s) = C(sI - A)^{-1}x(0) + \underbrace{(C(sI - A)^{-1}B + D)}_{H(s)}U(s) \end{aligned}$$

- since $(sI - A)^{-1} \to \mathcal{L}e^{tA}$, we can get:

$$x(t) = e^{tA}x(0) + e^{tA} * Bu(t) \quad \text{and} \quad y(t) = Ce^{tA}x(0) + \underbrace{(Ce^{tA}B + D\delta(t))}_{h(t)} * u(t)$$

where $H(s)$ and $h(t)$ are transfer matrix and impulse matrix, respectively

- discretization: if we sample the states and outputs with interval $h$ as $x_d(k) = x(kh)$ and $y_d(k) = y(kh)$, we can get:

$$\begin{aligned} x_d(k+1) &= x(kh+h) \\ &= e^{(kh+h)A}x(0) + e^{tA} * Bu(t)|_{t=kh+h} \\ &= e^{hA}\left(e^{khA}x(0) + e^{tA} * Bu(t)|_{t=kh}\right) \\ &\quad + \int_{t=0}^{h} e^{(h-\tau)A}Bu(\tau+kh)d\tau \\ &= e^{hA}x_d(k) + \int_{t=0}^{h} e^{(h-\tau)A}Bu(\tau+kh)d\tau \end{aligned}$$

and

$$\begin{aligned} y_d(k) &= Ce^{khA}x(0) + h(t) * u(t)|_{t=kh} \\ &= Cx_d(k) + Du(kh) \end{aligned}$$

# Example

when inputs are piecewise constant, $i.e.$, $u(t) = u_d(k)$ for $kh \leq t < (k+1)h$, rewrite the linear dynamic system in the discretized form

*Solution.* with the piecewise constant inputs, the above system of equations can be rewritten as:

$$\begin{aligned} x_d(k+1) &= e^{hA} x_d(k) + \int_{t=0}^{h} e^{(h-\tau)A} B u(\tau + kh) d\tau \\ &= \underbrace{e^{hA}}_{A_d} x_d(k) + \underbrace{\left( \int_{t=0}^{h} e^{\tau A} d\tau \right) B}_{B_d} u_d(k) \end{aligned}$$

and

$$y_d(k) = \underbrace{C}_{C_d} x_d(k) + \underbrace{D}_{D_d} u_d(k)$$