# About the region of convergence of the z-transform

The z-Transform of a sequence $f[n]$ is defined as $S(z) = \sum_{\infty}^{\infty} f[n]z^{-n}$, for those values of $z$[1] for which the infinite sum converges, such set of values of $z$ is called the Region of Convergence of the z-transform $S(z)$. This document describes the possible shapes the Region of Convergence (ROC) may take. We start by describing the ROC shape of one sided sequences from which we'll deduce the ROC shape for two sided sequences.

## Region of Convergence for z-transforms of Unilateral sequences

Let $f[n]$ be an anticausal sequence, i.e. $f[n] = 0$ for $n \geq 0$. Its z-transform is $S(z) = \sum_{n=-\infty}^{\infty} f[n]z^{-n} = \sum_{n=0}^{\infty} f[-n]z^{n}$. If the sequence is of finite duration, its z-transform is a finite polynomial with nonnegative powers of $z$, hence it converges for all finite values values of z and its ROC is the whole z-plane. On the other hand, if the sequence is of infinite duration, the situation is less straight forward and we'll need the following lemma:

**Lemma 1** *If a power series $\sum_{n=0}^{\infty} a_n z^n$ converges when $z = z_1$ ($z_1 \neq 0$), then it is absolutely convergent[2] at each point $z$ in the open disk $|z| < |z_1|$.*

The proof is as follows, we assume that the series $\sum_{n=0}^{\infty} a_n z^n$ converges, hence the terms $a_n z_1^n$ are bounded; that is,

$$|a_n z_1^n| \leq M \qquad (n = 0, 1, 2, \ldots)$$

for some positive constant $M$. If $|z| < |z_1|$ and we let $\rho$ denote the modulus $|z/z_1|$, we can see that

$$|a_n z^n| = |a_n z_1^n| \left|\frac{z}{z_1}\right|^n \leq M\rho^n \quad (n = 0, 1, 2, \ldots)$$

where $\rho < 1$. Since $\sum_{n=0}^{\infty} M\rho^n$ converges when $\rho < 1$, we can conclude that

$$\sum_{n=0}^{\infty} |a_n z^n| < \infty \quad \Longrightarrow \quad \sum_{n=0}^{\infty} a_n z^n \quad \text{converges}$$

in the open disk $|z| < |z_1|$.

This lemma tells us that the set of all points *inside* some circle centered at the origin is part of the region of convergence for the power series $\sum_{n=0}^{\infty} a_n z^n$, provided it converges at some point

[1] $z$ is a complex variable.
[2] a series $\sum_{n=0}^{\infty} a_n$ is said to converge absolutely if the series $\sum_{n=0}^{\infty} |a_n|$ converges

other than $z = 0$, what's more, it converges absolutely inside that circle of convergence. The greatest circle centered at the origin $|z| = R$ such that the series converges at each point inside is called the *circle of convergence* of the series. The series cannot converge at any point $z_2$ outside that circle, according to the theorem; for if it did, it would converge everywhere inside the circle centered at the origin and passing through $z_2$. The first circle could not then be the circle of convergence. Note that it may be that the series $\sum_{n=0}^{\infty} a_n z^n$ converges for all (finite) $z$, in that case we can think of the circle of convergence as being $|z| = \infty$ and say that the ROC is $|z| < \infty$ (This region is called the finite z-plane). Notice as well that the series may or may not converge at points at the circle of convergence $|z| = R$. We can summarize the consequences of Lemma 1 as follows:

*The Region of Convergence of the z-transform of an anticausal sequence is either $|z| < \infty$ or of the form*

$$|z| < R \quad \bigcup \quad B$$

*for some nonnegative constant $R$ and $B$ a subset*[3] *of the so-called circle of convergence $|z| = R$.*

We'll say that a sequence $f[n]$ is causal if $f[n] = 0$ for $n < 0$. Following a similar argument it can be shown that *the ROC of the z-transform of a causal sequence is the exterior of a circle (called the circle of convergence) possibly including points in the boundary*. In particular, if $f[n]$ is causal and we can find a positive integer $N$ such that $f[n] = 0$ for $n > N$, then the ROC will be $|z| > 0$, including $z = 0$ only if $f[n] = 0$ for $n \neq 0$, in which case, the ROC of the z-transform is the whole z plane[4].

## Region of Convergence for Bilateral sequences

In general, a sequence can be two-sided, in that case, the ROC of its z-transform is obviously the intersection of the ROC's corresponding to its causal and anticausal parts. We can summarize the results for the ROC of z-transforms as follows:

*The Region of Convergence of the z-transform of a sequence can have one of the following forms:*

- The whole z-plane
- The finite z-plane $|z| < \infty$
- A subset $B$ of a circle $|z| = R$
- The exterior of a circle $|z| > R_1 \quad \bigcup \quad B_1$
- An annulus$R_1 < |z| < R_2 \quad \bigcup B_1 \bigcup B_2$

where $R_1$ and $R_2$ are nonnegative constants, and $B_1$ and $B_2$ are subsets[5] of the circles $|z| = R_1$ and $|z| = R_2$ respectively.

[3]The subset $B$ can be the empty set.

[4]Note that the whole z-plane differs from the finite z-plane $|z| < \infty$ as a ROC in the sense that for the later case $\lim_{z \to \infty} S(z)$ does not exist

[5]Subsets $B_1$ and $B_2$ can be the empty sets.

For convenience, we say a region is an open annulus if it has one of the following forms: $|z| < R$, $|z| > R$, or $R_1 < |z| < R_2$, where $R$, $R_1$ and $R_2$ are nonnegative constants. From our previous discussion is easy to see that in the cases when the ROC of a z-transform includes an open annulus, the z-transform converges absolutely in that open annulus and what's more, it converges uniformly. From this it can be proved to be analytic in that open annulus.

## Region of convergence for rational functions of $z$

In many practical cases we can find a closed form expression $F(z)$ for the z-transform $S(z)$ of the sequence for any $z$ in the ROC of $S(z)$, i.e. $S(z) = F(z),\ z \in ROC$ where $F(z)$ does not have a infinite sum of terms. Most of the sequences we work with are such that their z-transform have a closed form expression $F(z)$ that is a rational function of $z$, i.e. $F(z) = \frac{N(z)}{D(z)}$ where $N(z)$ and $D(z)$ are finite order polynomials in $z$. In such cases we say that $F(z)$ has a pole[6] at $z_o$ if $\lim_{z \to z_o} F(z) = \infty$. Also, $F(z)$ is said to have a pole at $\infty$ if $\lim_{z \to \infty} F(z) = \infty$.

Using partial fraction expansion theory, it can be proved that the Region of Convergence of a z-tranform that has a rational function $F(z)$ as a closed form expression is either the whole z-plane, the finite z-plane or exactly an open annulus whose boundaries are circles that pass through at least one of the poles of $F(z)$, obviously such open annulus cannot contain any of the poles. Hence, *the ROC for rational functions of z can have one of the following forms:*

- The whole z-plane
- The finite z-plane $|z| < \infty$
- $|z| > |z_o|$
- $|z| < |z_o|$
- $|z_o| < |z| < |z_1|$

where $z_o$ and $z_1$ are poles of $F(z)$ and there is no other pole $z_2$ such that $|z_o| < |z_2| < |z_1|$. Notice the difference between these forms and the general possible forms for the ROC given before. We can see that the fact that the z-transform is a rational function of $z$, excludes the points at the circle of convergence as possible points of the ROC[7]. This fact allows us to easily determine the possible ROC's for a rational z-transform $F(z)$ by just finding its poles and subsequently the open annulus delimited by them . For each of these open annulus $F(z)$ will have a unique inverse z-transform.

[6] the definition of a pole for general functions of z is more elaborated and requires the use of Laurent series.

[7] See section 3.2 of Discrete-Time Signal Processing from Oppenheim and Schafer, Second edtion, for a summary of the properties of the ROC of z-transforms with rational closed form.