1. Colorful coins

We are given three coins. The first coin is a fair coin painted blue on the heads side and white on the tails side. The other two coins are biased so that the probability of heads is $p$. They are painted blue on the tails side and red on the heads side. One coin is randomly chosen and flipped twice.

(a) Describe the outcomes in the sample space, and give their probabilities. [NOTE: You may want to draw a tree to illustrate the sample space.]

(b) Now suppose two coins are chosen randomly with replacement and each flipped once. Describe the outcomes in the sample space in this new experiment, and give their probabilities. Are they the same as in part (a)? [NOTE: You may want to draw a tree to illustrate the sample space.]

(c) Now suppose two coins are chosen randomly without replacement and each flipped once. Describe the outcomes in the sample space in this new experiment, and give their probabilities. Are they the same as in parts (a) or (b)? [NOTE: You may want to draw a tree to illustrate the sample space.]

(d) Suppose the probability that the two sides that land face up are the same color is $\frac{29}{96}$ in the experiment in part (c). What does this tell you about the possible values of $p$?

(e) Let $A$ be the event that you get a head on the first flip and $B$ is the event that you get a head on the second flip. In each of the experiments in (a), (b) and (c), verify if $A$ and $B$ are independent events.

2. Beat Deep Blue

You’ve been booked to play a chess tournament where to win the tournament, you have to win two consecutive games of chess, out of three games. You have the choice of playing David Tse, then Deep Blue, then David Tse (TBT)—or Deep Blue, then David Tse, then Deep Blue (BTB). David Tse is a lousy chess player; Deep Blue is very difficult to beat. Which schedule should you choose, to maximize your chances of winning the tournament? (NOTE: If you win the first and third game of chess but lose the second, you lose the tournament!)

3. Conditional probability

a) I have a bag containing either a $1 or $5 bill (with probability $1/2$ for each of these two possibilities). I then add a $1 bill to the bag, so it now contains two bills. The bag is shaken, and you randomly draw a bill from the bag (without looking). Suppose it turns out to be a $1 bill. If a second student draws the remaining bill from the bag, what is the chance that it too is a $1 bill?

b) Your gambling buddy found a website online where he could buy trick coins that are heads or tails on both sides. He puts three coins into a bag: one coin that is heads on both sides, one coin that is tails on both sides, and one that is heads on one side and tails on the other side. You shake the bag, draw out a coin at random, put it on the table without looking at it, then look at the side that is showing. Suppose you notice that the side that is showing is heads. What is the probability that the other side is heads?

4. Being discreet

A non-profit wants to poll a sample of people to ask them whether they have ever had an extramarital affair. This being an extremely sensitive subject, one obvious problem is that if the surveyors ask this question straight-out, respondents may lie to avoid revealing personal information about their private lives.
The surveyors come up with the following clever scheme. They will ask the respondent to secretly roll a fair die. If the die comes up 1, 2, 3, or 4, the respondent is supposed to answer truthfully. If the die comes up 5 or 6, the respondent is supposed to answer the opposite of the truthful answer. The respondent is cautioned not to reveal what number came up on the die. Notice that if the respondent answers “Yes,” this answer is not necessarily incriminating: for all the surveyer knows, this particular respondent might have rolled a 5 or 6 and might have never had an affair in his/her life.

Let $p$ be the probability that, if we select a person at random, then they will have had an extramarital affair. (Of course, the surveyors do not know $p$; that is what they want to estimate.) Let $q$ denote the probability that, if we select a person at random and have them follow the scheme above, then they will answer “Yes.”

a) Calculate a simple formula for $q$, as a function of $p$.

b) Suppose the surveyors have surveyed 1000 people. What is a sensible way of estimating $q$?

c) Next, suppose the surveyors have estimated $q$. Now they want to solve for $p$. Find a simple formula for $p$, as a function of $q$.

5. Communication Channel.

The probability transition diagram for a ternary communication channel is shown in the following figure.

Assume the input symbols 0, 1, and 2 are sent with probabilities 1/2, 1/4, and 1/4, respectively.

a) Find the probabilities of the output symbols.

b) Given that 1 is received, find the probabilities that the input was 0, 1, or 2.

Your answers should be in terms of the conditional error probability $e$.

6. Serve on the jury

In the OJ Simpson murder trial, OJ Simpson was accused of murdering his ex-wife, Nicole Simpson. The prosecution introduced evidence showing that OJ had previously abused Nicole. One of Simpson’s defense lawyers, Alan Dershowitz, made the following argument in OJ Simpson’s defense. Dershowitz stated that 1 in 1,000 women abused by their husbands are later killed by their abuser, so the fact that OJ Simpson had previously abused his wife is not relevant and should be disregarded. Assume for this problem that Dershowitz’s 1 in 1,000 statistic is accurate.

a) Are we entitled to conclude that there is only a 1/1000 probability that OJ Simpson murdered Nicole? Why or why not?

b) Suppose we select at random a woman who has been abused by her husband. Define the following events: $M$ is the event that the woman is murdered at some point in her life; $G$ is the event that the woman is murdered by her abuser at some point in her life. A plausible estimate is that 0.2% of abused women will be murdered by someone other than their abuser at some point in their life. Calculate the probability that the selected woman is murdered by her abuser, given that she is murdered.
c) Based upon your answer to part (b), do you agree or disagree with Dershowitz’s argument? Based upon your calculation, would you consider it relevant that OJ Simpson previously abused Nicole? Would you judge it more accurate to use the 1 in 1,000 number or the number you calculated in part (b)? Why?