Lecture 14

Reading: 3.6.1 & 3.6.2 Galligas

Today
- KF wrap up
- Robust estimation
- Intro to random process

Receup

System
\[
\begin{align*}
X_1 &\sim N(\bar{X}, \sigma^2_{x_1}) \\
X_{n+1} &= \alpha X_n + W_n \\
Y_n &= X_n + Z_n
\end{align*}
\]

Filter
\[
\begin{align*}
\hat{X}_1 &= \bar{X} + \frac{\sigma^2_{x_1}}{\sigma^2_{x_1} + \sigma^2} (Y_1 - \bar{X}) \\
\hat{X}_{n+1} &= \hat{X}_n + \frac{2}{\sigma^2_{x_1} + \sigma^2} (Y_{n+1} - \hat{X}_n)
\end{align*}
\]

\[
\begin{align*}
\hat{X}_n &= \alpha X_n + \frac{2}{\sigma^2_{x_1} + \sigma^2} (Y_n - \hat{X}_n - \alpha \hat{X}_n)
\end{align*}
\]

\[
\frac{1}{V_{n+1}} = \frac{1}{\alpha^2 V_n + \sigma^2} + \frac{1}{\sigma^2}
\]

\[
\alpha X_3 = \text{prediction of } X_2 \text{ based on } Y_1, Y_2, Y_3
\]

\[
\begin{align*}
X_1 - X_2 - X_3 - X_4 \\
Y_1 \quad Y_2 \quad Y_3 \quad Y_4
\end{align*}
\]
\[ U_n^2 = V_n^2 \]
\[ V_n^2 = E \left[ (x_n - \hat{x}_n)^2 \right] \quad \text{based on } Y_1, \ldots, Y_n \]
\[ U_n^2 = E \left[ (x_{\text{true}} - \hat{x}(Y_1, \ldots, Y_n))^2 \right] \]

\[ n \to \infty \quad \text{what happens to } U_n \text{ and } V_n \]

\[ V_n \to V_{\text{opt}} \quad \text{where} \quad V_{\text{opt}}^2 = \frac{1}{\lambda^2 V_{\text{opt}}^2 + 1} \quad \text{for } n = 1, 2, 3, \ldots \]

"Riccati equation"

This relation does not depend on \( \sigma_{x_1}^2 \): in the long term, the estimation error does not depend on the prior distribution.

In the limiting case, \( \hat{x}_{n+1} = \lambda \hat{x}_n + \frac{U_0}{U_{\infty} + \sigma_n^2} (Y_{n+1} - 2 \lambda \hat{x}_n) \)

In the long run, we have a LTI system and a LTI filter.
We can do a frequency domain linear time invariant analysis.
Robust estimation.

Let $X, Y$ be jointly distributed r.v.s (not necessarily Gaussian)

$$E(x) = E(Y) = 0$$

Goal: minimize $E \left[ (x - \hat{X}(Y))^2 \right]$: linear least square estimation.

$\hat{X}(Y) = BY$ suppose we restrict ourselves to only using the linear estimator.

1. What is the best $B$?
2. Does this estimator have any good property?

$\hat{X}_{\text{MML}}(Y) = E(X | Y)$

Restate the problem here: minimize $\min_B E \left[ (X - BY)^2 \right]$

Claim: we already have the solution to this problem.

$$B^* = \frac{\text{cov}(X,Y)}{\sigma_Y^2}$$

If $(X,Y)$ were jointly Gaussian, then $E[X | Y] = \frac{\text{cov}(X,Y)}{\sigma_Y^2} Y$

$X = \alpha^* Y + Z$
\[
E \left[ (X - \beta^* Y)^2 \right] \\
= E \left[ X^2 - 2 \beta X Y + \beta^2 Y^2 \right] \\
= E[X^2] - 2 \beta E[XY] + \beta^2 E[Y^2]
\]

This problem depends on the joint distribution only through its second moments. So we can pretend that the joint distribution is actually Gaussian with the same second moments. Then the MMSE estimator gives the solution to the problem.

Objective fn depends only on second-order moments of \( f_{X,Y} \)
\[ \Rightarrow \beta^* \text{ depends only } \]
\[ \Rightarrow \text{ "pretend" that } X, Y \text{ is jointly Gaussian and solve the MMSE problem.} \]

\underline{Good property}

Only know the second-order moments of \( f_{X,Y} \)

\[ \text{LSE : } \hat{X}(Y) = \beta^* Y \]

will minimize the worst-cost error over all such joint distribution
LLSE = MMSE. At this particular point, LLSE has the smallest error consistent with 2nd order moment.