Reading "Random Processes" by A.E. Gamal (on course site)

Today:
- Examples of random processes
- Stationarity

Progression:

random variable → multiple rvs → infinite # of rvs

undergrad  → random vector → random process

random vectors = probability + linear algebra
random process = probability + signal & systems

e.g. $X_1, X_2, X_3, \ldots$

1) iid random variables

2) random walk:

$$X_0 = 0, \quad N_n \text{ i.i.d. } \pm 1 \text{ w.p. } \frac{1}{2} \text{ each}$$

$$X_n = X_{n-1} + N_n$$
At time $n$: distribution of $X_n$?

$X_n = \sum_{i=1}^{n} \alpha W_i$

$\text{var}(X_n) = n$

$\text{std}(X_n) = \sqrt{n} \implies \frac{X_n}{\sqrt{n}} \sim N(0,1)$

$P(X_n = n) = 2^{-n}$

e.g. $\{X_n\}$ a Markov chain

"Mickey Mouse Markov Chain"

$X_0 = 0$ or $P_r \{X_0 = 0\} = \frac{1}{3} \quad P_r \{X_0 = 1\} = \frac{2}{3}$

initial distribution of states
Complete specification of random process

given a collection of $X_0, X_1, \ldots, X_k$.

I can in principle calculate the joint distribution

E.g., $X_0, X_1, X_2, X_3$

$$P_r\{X_0 = 0, \; X_1 = 1, \; X_2 = 1, \; X_3 = 0\}$$

$$= P_r\{X_0 = 0\} \cdot P_r\{X_1 = 1 | X_0 = 0\} \cdot P_r\{X_2 = 1 | X_1 = 1\} \cdot P_r\{X_3 = 0 | X_2 = 1\}$$

$$= \frac{1}{3} \alpha (1-\alpha) \cdot \alpha$$

E.g., $X_0 = 0$: $\alpha = 0$

$$X_n = \alpha X_{n-1} + W_n \quad \text{Where i.i.d. } N(0, \sigma^2)$$

$$E[X_n] = 0 \quad \text{for all } n$$

$\alpha > 1$

$\alpha = 1$

$\alpha < 1$
\[
\text{var}(x_{n+1}) = \alpha^2 \text{var}(x_n) + \sigma_w^2
\]

\[
\text{var}(x_{n+1}) = \alpha^2 V_n + \sigma_w^2
\]

\[
V_0 = 0
\]

\[
V_1 = \alpha^2 \sigma_w^2
\]

\[
V_2 = \alpha^2 \sigma_w^2 + \sigma_w^2
\]

\[
V_3 = \alpha^2 \left( \alpha^2 \sigma_w^2 + \sigma_w^2 \right) + \sigma_w^2
\]

\[
= \left( \alpha^4 + \alpha^2 + 1 \right) \sigma_w^2
\]

\[
V_n = \sigma_w^2 \sum_{i=0}^{n-1} \alpha^{2i} \sim \alpha^{2n} \sigma_w^2
\]

\[
\alpha > 1 \Rightarrow \text{explode}
\]

\[
\alpha = 1 \Rightarrow \text{random walk} \quad V_n = n \sigma_w^2 \quad X_n \sim N(0, n \sigma_w^2)
\]

\[
\alpha < 1 \quad V_n \rightarrow \frac{1}{1 - \alpha^2} \sigma_w^2
\]

\[
\text{Stationarity:}
\]

\[
\text{"dist" remains the same over time.}
\]

\[
X_1, X_2, X_3, \ldots
\]

\[
\text{dist of } X_n = f_n \text{ is the same for all } n.
\]
e.g., \( X_1, X_2, \ldots, X_n \)

2) \( \text{dist of } X_1, X_2 = \text{dist of } X_2, X_3 \\
    = \text{dist of } X_3, X_4 \)

3) \( \text{dist of } X_1, X_2, X_3 = \text{dist of } X_3, X_4, X_5 \\
    = \text{dist of } X_4, X_5, X_6 \ldots \)

\( X_1, X_2, X_3, X_4, \ldots \)

The distribution remain the same instead of the values

Def: \( \{X_n\} \) is stationary if \( \forall n \in \mathbb{N}, \quad X_1, \ldots, X_n \) has the same distribution as \( (X_{n+k}, \ldots, X_{n+k+k}) \)

Ex: Stationarity.
  i) i.i.d. \( \checkmark \)
  ii) random walk \( \Rightarrow \) not a stationary process \( X \)
  iii) \( \text{MMMCM} \Rightarrow X \) in general

Mickey Mouse \( \checkmark \) if we start at stationarity.

\[
\begin{align*}
\Pr\{X_0 = 0\} &= \pi_0 \\
\Pr\{X_1 = 1\} &= 1 - \pi_0 \\
\Pr\{X_1 = 0\} &= (1 - \alpha) \pi_0 + \alpha \left(1 - \pi_0\right) \\
\Pr\{X_0 = 1\} &= \pi_0 + \alpha - 2\alpha \pi_0 = \pi_0 (1 - 2\alpha) + \alpha \\
\Pr\{X_1 = 1\} &= \pi_0 \alpha + (1 - \pi_0) (1 - \alpha)
\end{align*}
\]
\[ P = \begin{bmatrix} 1 - \alpha & \alpha \\ \alpha & 1 - \alpha \end{bmatrix} \]

\[
\begin{bmatrix}
\Pr(X_1 = 0) \\
\Pr(X_1 = 1)
\end{bmatrix} = P
\begin{bmatrix}
\pi_0 \\
1 - \pi_0
\end{bmatrix}
\]

Actually, if \( \pi_0 = \frac{1}{2} = 1 - \pi_0 \), \( \Pr(X_1 = 0) = \frac{1}{2} = \Pr(X_1 = 1) \)

Does \((x_0, x_1)\) have the same dist as \((x_1, x_2)\)?

\[
P_{x_0 x_1} = P_{x_0} \cdot P_{x_1 | x_0} \quad \Rightarrow \quad \text{As long as } P_{x_0} = P_{x_1} \text{, we have } P_{x_0 x_1} = P_{x_1 x_2}
\]

**Is the following process stationary?**

\[
X_{n+1} = \alpha X_n + W_n
\]

\[
x_0 = 0 \quad \text{No because } x_0 \text{ is a number and other } x_i \text{'s are R.V.}
\]

**Question:** can we change the distribution of the initial state to make the process stationary?