1. Exercise 10.2 in Gallager.

2. Consider the estimation problem:

\[
\begin{align*}
Y_1 &= X + Z_1 \\
Y_2 &= X + Z_2
\end{align*}
\]

where \( X, Z_1, Z_2 \) are mutually independent Gaussian random variables. \( X \) has mean \( \bar{X} \) and variance \( \sigma_0^2 \), while \( Z_1, Z_2 \) have mean 0 and variance 1.

(a) Compute the conditional distribution of \( X \) given \( Y_1 = y_1 \).

(b) Compute the conditional distribution of \( (X, Z_1) \) given \( Y_2 = y_2 \).

(c) Using (a) and (b) compute the conditional distribution of \( X \) given \( Y_1 = y_1 \) and \( Y_2 = y_2 \) in a successive fashion by first conditioning on \( Y_1 = y_1 \) and then on \( Y_2 = y_2 \).

(d) Check that your answer in part (c) is the same as what you would have got by computing the conditional distribution directly.

3. Consider the MMSE estimation problem with a \( n \)-dimensional observation:

\[
Y = hX + Z,
\]
where \( X \sim \mathcal{N}(0, 1) \) and \( Z \sim \mathcal{N}(0, \sigma^2 I) \) and independent of \( X \).

(a) Show that \( V = h^T Y \) is a sufficient statistic for the estimation problem. Recall that a sufficient statistic \( V = v(Y) \) is such that \( X, V, Y \) forms a Markov chain.

(b) Does \( V \) remain a sufficient statistic if \( X \) is non-Gaussian?

4. Let \( (X, Y) \) be two zero-mean jointly Gaussian random variables with non-zero variances.

In lecture we derive the conditional distribution of \( X \) given \( Y \) by writing \( X = \alpha^* Y + Z \) where \( Z \) is independent of \( Y \), zero-mean and Gaussian. This implies that conditional on \( Y \), \( X \) is Gaussian with mean \( \alpha^* Y \).

(a) What is the choice of \( \alpha^* \) in terms of the statistics of \( X \) and \( Y \)?

(b) Suppose \( X \) and \( Y \) are both zero-mean but not jointly Gaussian. Write \( X = \alpha^* Y + Z \) for the same \( \alpha^* \) as in part (a). Suppose \( Z \) has density \( f \). Is \( Z \) zero-mean? Conditional on \( Y \), can we still say that \( X \) has the same density as \( Z \) but shifted by \( \alpha^* Y \)? Explain.
(c) Now back to the jointly Gaussian $X$ and $Y$. Instead of using the representation $X = \alpha Y + Z$, derive the conditional density of $X$ given $Y$ explicitly.

(d) Now suppose $X$ and $Y$ are zero-mean random vectors and they are jointly Gaussian and $K_Y$ is full rank. Compute from first principles the conditional density of $X$ given $Y$. You can extend either one of the two approaches for the scalar case.

5. Exercise 10.3 in Gallager.