1. Change of probability measure  The random variable $X$ is uniformly distributed in $[0, 1]$. What is the density function of $Y = X^2$?

2. Constructing covariance matrix  $\Sigma$ is a $n \times n$ matrix with entries $\Sigma_{ij} = \min\{i, j\}$. Prove this is a covariance matrix by describing a construction of an $n$ dimensional random vector with covariance matrix $\Sigma$.
   Hint: Start with $n$ iid zero mean, unit variance random variables.

3. Gaussian Random Vectors.  Let $X \sim \mathcal{N}(0, K_X)$ with

   $$K_X = \begin{bmatrix} a & b_1 & c \\ b_1 & a & b_2 \\ c & b_2 & a \end{bmatrix},$$

   where $a, b_1, b_2, c \neq 0$, $b_1^2 < a^2$, and $b_2^2 < a^2$.

   (a) Specify the joint pdf of $X_1 + X_2$ and $X_1 - X_2$. Are they statistically independent?

   (b) Find a linear transformation $A$ such that

   $$Y = AX \sim \mathcal{N}(0, \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}).$$

   Your answer should be in terms only of the entries of $K_X$. Hint: Note that

   $$\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$$

4. Two Coins  You are given two coins, coin 1 has bias $1/2$ and coin 2 has a randomly selected bias $P \sim \mathcal{U}[0, 1]$. You pick one of them at random and flip it twice. Let $X = 1$ if coin 1 is selected and $X = 2$ if coin 2 is selected with $p_X(1) = p_X(2) = 1/2$. Let $Y_i = 1$ if the outcome of flip $i$ is heads and $Y_i = 0$ if the outcome is tails for $i = 1, 2$. Observing the outcomes of these two coin flips $y_1$ and $y_2$, you wish to decide which coin was selected. Assume that $Y_1$ and $Y_2$ are conditionally independent given the value of the bias of the selected coin.

   (a) Find the estimate $\hat{x}(y_1, y_2) \in \{1, 2\}$ that minimizes the probability of error $\mathbb{P}\{X \neq \hat{x}\}$. Your answer should be explicit in terms only of $y_1$ and $y_2$.

   (b) Find the minimum probability of error.