1. **Noise cancellation**  A classic problem in statistical signal processing involves estimating a weak signal (e.g., the heart beat of a fetus) in the presence of a strong interference (the heart beat of its mother) by making two observations—one with the weak signal present and one without (by placing one microphone on the mother’s belly and another close to her heart). The observations can then be combined to estimate the weak signal by “canceling out” the interference. The following is a simple version of this application.

Let the weak signal $X$ be a random variable with mean $\mu$ and variance $\sigma^2$. Let the observations be $Y_1 = X + Z_1$ and $Y_2 = Z_1 + Z_2$, where $Z_1$ is the strong interference and $Z_2$ is measurement noise. Assume that $Z_1$ and $Z_2$ are zero mean with variances $\sigma^2_1$ and $\sigma^2_2$, respectively. Further assume that $X$, $Z_1$, and $Z_2$ are uncorrelated. Find the LLSE estimate of $X$ given $Y_1$ and $Y_2$ and the corresponding MSE. Interpret the results.

2. **Singular Covariance Matrix**. From the book we know the general optimal linear estimator of $X$ given vector $Y$ is given by:

$$\hat{X}_{\text{lin}}(Y) = K_{XY}K_{YY}^{-1}(Y - \bar{Y}) + \bar{X},$$

(1)

where $\bar{Y}$ and $\bar{X}$ are the mean of $Y$ and $X$.

However, this formula would not work if the covariance matrix $K_Y$ is not invertible. In this problem, you are required to investigate this situation and obtain the corresponding solutions.

(a) Prove that if the covariance matrix of a $d$-dimensional random vector $Y$ is not invertible, then there must exist at least one component of $Y$ which is expressible as a linear combination of the others.

(b) Suppose $X$ is a zero-mean scalar random variable, $Y$ is a zero-mean random vector with covariance matrix

$$K_Y = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix},$$

(2)

and the covariance between $X$ and $Y$ is

$$K_{XY} = [3, 4, 7].$$

(3)

Derive the optimal linear estimator of $X$ based on $Y$.

3. **Additive Non-White Gaussian Noise Channel** Let $Y_i = X + Z_i$ for $i = 1, 2, \ldots, n$ be $n$ observations of a signal $X \sim \mathcal{N}(0, \sigma^2)$. The additive noise random variables $Z_1, Z_2, \ldots, Z_n$ are zero mean jointly Gaussian random variables that are independent of $X$ and have correlation $E(Z_iZ_j) = N \cdot 2^{-|i-j|}$ for $1 \leq i, j \leq n$. Find the best MSE estimate of $X$ given $Y_1, Y_2, \ldots, Y_n$, and its MSE. Hint: the coefficients for the best estimate are of the form $h^T = [a \ b \ b \ \cdots \ b \ b \ a]$. 
