1. **Stationary Gauss-Markov process.** Consider the following variation on the Gauss-Markov process

\[ X_0 \sim \mathcal{N}(0, a) \]

\[ X_n = \frac{1}{2} X_{n-1} + Z_n, \quad n \geq 1, \]

where \( Z_1, Z_2, Z_3, \ldots \) are i.i.d. \( \mathcal{N}(0, 1) \) independent of \( X_0 \). Find \( a \) such that \( X_n \) is stationary.

2. **Moving-average process.** Let \( \{X_n : n \geq 1\} \) be a discrete-time white Gaussian noise process, that is, \( X_1, X_2, X_3 \ldots \) are i.i.d. random variables with \( X_n \sim \mathcal{N}(0, N) \). Consider the moving-average process \( \{Y_n : n \geq 2\} \) defined by

\[ Y_n = \frac{2}{3} X_{n-1} + \frac{1}{3} X_{n-2}, \quad n \geq 2. \]

Let \( X_0 = 0 \). Find the mean and covariance functions for the process \( Y_n \).

3. **Absolute value random walk.** Let \( X_n \) be a random walk defined by

\[ X_0 = 0, \quad X_n = \sum_{i=1}^{n} Z_i, \quad n \geq 1, \]

where \( \{Z_i\} \) is an i.i.d. process with \( \mathbb{P}\{Z_1 = -1\} = \mathbb{P}\{Z_1 = +1\} = \frac{1}{2} \). Define the absolute value random process \( Y_n = |X_n| \).

(a) Find \( \mathbb{P}\{Y_n = k\} \).

(b) Find \( \mathbb{P}\{\max\{Y_i: 1 \leq i < 20\} = 10 \mid Y_{20} = 0\} \).

4. **Discrete-time LTI system with white noise input.** Let \( \{X_n : -\infty < n < \infty\} \) be a discrete-time white noise process, i.e., \( \mathbb{E}(X_n) = 0 \) and

\[ K_X(m) = \begin{cases} 
1 & m = 0 \\
0 & \text{otherwise}
\end{cases} \]

The process is filtered using a linear time invariant system with impulse response

\[ h(n) = \begin{cases} 
\alpha & n = 0 \\
\beta & n = 1 \\
0 & \text{otherwise}
\end{cases} \]

Find \( \alpha \) and \( \beta \) such that the output process \( Y_n \) has

\[ K_Y(m) = \begin{cases} 
2 & m = 0 \\
1 & |m| = 1 \\
0 & \text{otherwise}
\end{cases} \]