Lecture 10 - Optimization

• LP
  – Process plants - Refineries
  – Actuator allocation for flight control
  – More interesting examples

• Introduce QP problem

• More technical depth
  – E62 - Introduction to Optimization - basic
  – EE364 - Convex Optimization - more advanced
Real-time Optimization in Control

• Important part of multivariable control systems
• Many actuators, control handles
• Quasistatic control, dynamics are not important
  – slow process
  – low-level fast control loops
  – fast actuators
Optimization methods

• Need to state problem such that a solution can be computed quickly, efficiently, reliably

• Least squares - linear quadratic problems
  – analytical closed form, matrix multiplication and inversion

• Linear Programming
  – simplex method

• Quadratic Programming
  – interior point

• Convex optimization: includes LP, QP, and more
Optimization in Process Plants
Optimization in Process Plants
Linear programming

• LP Problem:

\[ Ax \leq b \]
\[ Gx = h \]
\[ J = f^T x \rightarrow \text{min} \]

\[ x \leq y \iff \begin{bmatrix} x_1 \leq y_1 \\ \vdots \\ x_n \leq y_n \end{bmatrix} \]

• Might be infeasible! … no solution satisfies all constraints

• Matlab Optimization Toolbox: \texttt{LINPROG}
Linear programming

\[ Ax \leq b \]
\[ Gx = h \]
\[ J = f^T x \rightarrow \text{min} \]

- Simplex method in a nutshell:
  - check the vertices for value of \( J \), select optimal
  - issue: exponential growth of number of vertices with the problem size
  - Need to do 10000 variables and 500000 inequalities.

- Modern interior point methods are radically faster
  - no need to understand, standard solvers are available
Refinery Optimization

• Crude supply chain - multiple oil sources
• Distillation - separating fractions
• Blending - ready products, given octane ratings
• Objective function - profit
• LP works ideally:
  – linear equalities and inequalities, single linear objective function
Blending Example

- A Blending Problem: A refinery produces two grades of fuel, A and B, which are made by blending five raw stocks of differing octane rating, cost and availability.

<table>
<thead>
<tr>
<th>Gasoline</th>
<th>Octane Rating</th>
<th>Price $/B</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>93</td>
<td>37.5</td>
</tr>
<tr>
<td>B</td>
<td>85</td>
<td>28.5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Stock</th>
<th>Octane Rating</th>
<th>Price $/B</th>
<th>Availability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>70</td>
<td>9.0</td>
<td>2000</td>
</tr>
<tr>
<td>2</td>
<td>80</td>
<td>12.5</td>
<td>4000</td>
</tr>
<tr>
<td>3</td>
<td>85</td>
<td>12.5</td>
<td>4000</td>
</tr>
<tr>
<td>4</td>
<td>90</td>
<td>27.5</td>
<td>5000</td>
</tr>
<tr>
<td>5</td>
<td>99</td>
<td>27.5</td>
<td>3000</td>
</tr>
</tbody>
</table>
Blending Example

SA1, SB1

SA2, SB2

SA3, SB3

SA4, SB4

SA5, SB5

1

2

3

4

5

1

2

3

4

5

A

B

US1

US2

US3

US4

US5

FA

FB
Blending Example

- LP problem formulation:

\[ J = 9US1 + 12.5US2 + 12.5US3 + 27.5US4 + 27.5US5 + 37.5FA + 28.5FB \rightarrow \text{MAX} \]

[Stock Availability]

\[
\begin{align*}
S1A & + S1B & + US1 & = 2000 \\
S2A & + S2B & + US2 & = 4000 \\
S3A & + S3B & + US3 & = 4000 \\
S4A & + S4B & + US4 & = 5000 \\
S5A+ & S5B & + US5 & = 3000 
\end{align*}
\]

[Fuel Quantity]

\[
\begin{align*}
S1A+S2A+S3A+S4A+S5A & = FA \\
S1B+S2B+S4B+S5B & = FB
\end{align*}
\]

[Fuel Quality]

\[
\begin{align*}
70S1A & + 80S2A & + 85S3A & + 90S4A & + 99S5A & \geq 93FA \, \text{[Quality A]} \\
70S1B & + 80S2B & + 85S3B & + 90S4B & + 99S5B & \geq 85FB \, \text{[Quality B]}
\end{align*}
\]

[Nonnegativity]

\[
\]
Matlab code for the example

% OctRt  Price $/B
Gas = [93  37.5;
     85  28.5];
%Stock  OctRt  Price $/B  Availability
Stock = [70   12.5       2000;
       80   12.5       4000;
       85   12.5       4000;
       90   27.5       5000;
       99   27.5       3000];

% Revenue
f = [zeros(10,1); Stock(:,3); Gas(:,2)];
% Equality constraint
G = [eye(5,5)  eye(5,5)  eye(5,5)  zeros(5,2);
     ones(1,5)  zeros(1,5)  zeros(1,5)  -1  0;
     zeros(1,5)  ones(1,5)  zeros(1,5)  0  -1];
h = [Stock(:,3); zeros(2,1)];
% Inequality (fuel quality) constraints
A = [-[Stock(:,1)'  zeros(1,5)  zeros(1,5);
       zeros(1,5)  Stock(:,1)'  zeros(1,5)]  diag(Gas(:,1))];
b = zeros(2,1);
% X=LINPROG(f,A,b,Aeq,beq,LB,UB)
x = linprog(-f,A,b,G,h,zeros(size(f)),[]);
Revenue = f'*x
Blending Example - Results

- Blending distribution:

  Produced Fuel:  
  A  2125  
  B  15875  

  Total Revenue:  $532,125
GPS

- Determining coordinates by comparing distances to several satellites with known positions
- See E62 website:
  http://www.stanford.edu/class/engr62e/handouts/GPSandLP.ppt
Computing Resource Allocation

- Web Server Farm
- LP formulation for the optimal load distribution
Aircraft actuator allocation

- Multiple flight control surfaces
Aircraft actuator allocation

- Multiple flight control surfaces: ailerons, elevons, canard foreplanes, trailing and leading edge flaps, airbrakes, etc

\[
\begin{bmatrix}
M_{\text{roll}} \\
M_{\text{pitch}} \\
M_{\text{yaw}}
\end{bmatrix} = B(\alpha, \varphi, V)u
\]

\[F = Bu\]

\[F_{\text{desired}} \xrightarrow{\text{Algorithm}} u\]
Actuator allocation

- Simplest approach - least squares

\[ u = B^\dagger F \]

\[ B^\dagger = (B^T B)^{-1} B^T \quad \text{solves} \quad Bu = F, \quad \|u\|_2^2 \rightarrow \text{min} \]

- LP optimization approach

\[ Bu = F, \quad \|w^T u\|_1 \rightarrow \text{min} \]

\[ \|w^T u\|_1 = \sum w_k \cdot |u_k|, \quad w_k \geq 0 \]

\[ \begin{array}{l}
\begin{align*}
w^T u^+ + w^T u^- & \rightarrow \text{min} \\
u^+ & \geq 0 \\
u^- & \geq 0 \\
Bu^+ - Bu^- & = F
\end{align*}
\end{array} \]

Solve the LP, get \( u = u^+ - u^- \)
Actuator allocation

- Need to handle actuator constrains (v - scale factor)

\[ \|w^T u\|_1 - v \rightarrow \min \]
\[ u^l \leq u \leq u^u \]
\[ Bu = vF \]
\[ 0 \leq v \leq 1 \]

- LP can be extended to include actuator constrains

\[ w^T u^+ + w^T u^- - v \rightarrow \min \]
\[ Bu^+ - Bu^- - vF = 0 \]
\[ u^l \leq u^+ \leq u^u \]
\[ u^l \leq -u^- \leq u^u \]
\[ 0 \leq v \leq 1 \]
Actuator allocation example

- Problem:
  \[ \| w^T u \|_1 - v \rightarrow \min \]
  \[ Bu = vF \]
  \[ B = \begin{bmatrix} 0.9 & -0.7 & 0.4 & 0.1 \end{bmatrix} \]
  \[ w = \begin{bmatrix} 0.1 & 0.1 & 0.02 & 0.001 \end{bmatrix} \]
  \[ -1 \leq u \leq 1 \]

- LP problem solution for \( F = 1.5 \)
Actuator allocation example

- LP problem solution for $F$ from -2.5 to 2.5
Extreme actuator allocation

- (Xerox) PARC jet array table
- Jets must be allocated to achieve commanded total force and torque acting on a paper sheet

\[
F = \sum \vec{f}_k
\]

\[
T = \sum \vec{f}_k \times \vec{r}_k
\]
Actuator allocation

• Least squares + actuator constraint

\[ Bu = F, \]
\[ \|u\|^2 \rightarrow \text{min} \]
\[ u^l \leq u \leq u^u \]

• This is a QP optimization problem
Quadratic Programming

• QP Problem:

\[ Ax \leq b \]

\[ Gx = h \]

\[ J = \frac{1}{2} x^T H x + f^T x \rightarrow \text{min} \]

• Matlab Optimization Toolbox: \textbf{QUADPROG}

• Same feasibility issues as for LP

• Fast solvers available

• More in the next Lecture...