Lecture 10 - Optimization

- LP
  - Process plants - Refineries
  - Actuator allocation for flight control
  - More interesting examples
- Introduce QP problem

- More technical depth
  - E62 - Introduction to Optimization - basic
  - EE364 - Convex Optimization - more advanced

Real-time Optimization in Control

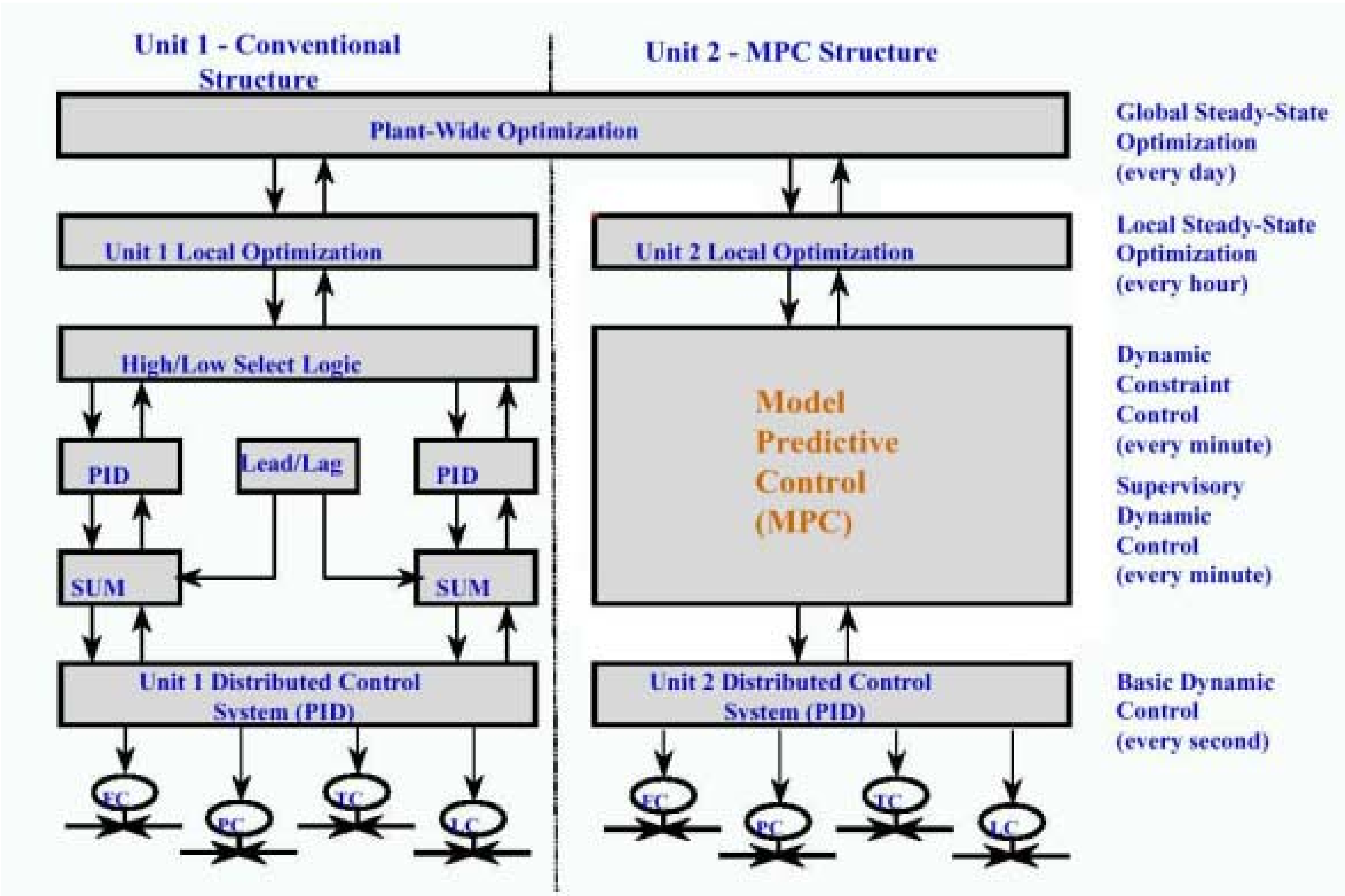
- Important part of multivariable control systems
- Many actuators, control handles
- Quasistatic control, dynamics are not important
  - slow process
  - low-level fast control loops
  - fast actuators

Optimization methods

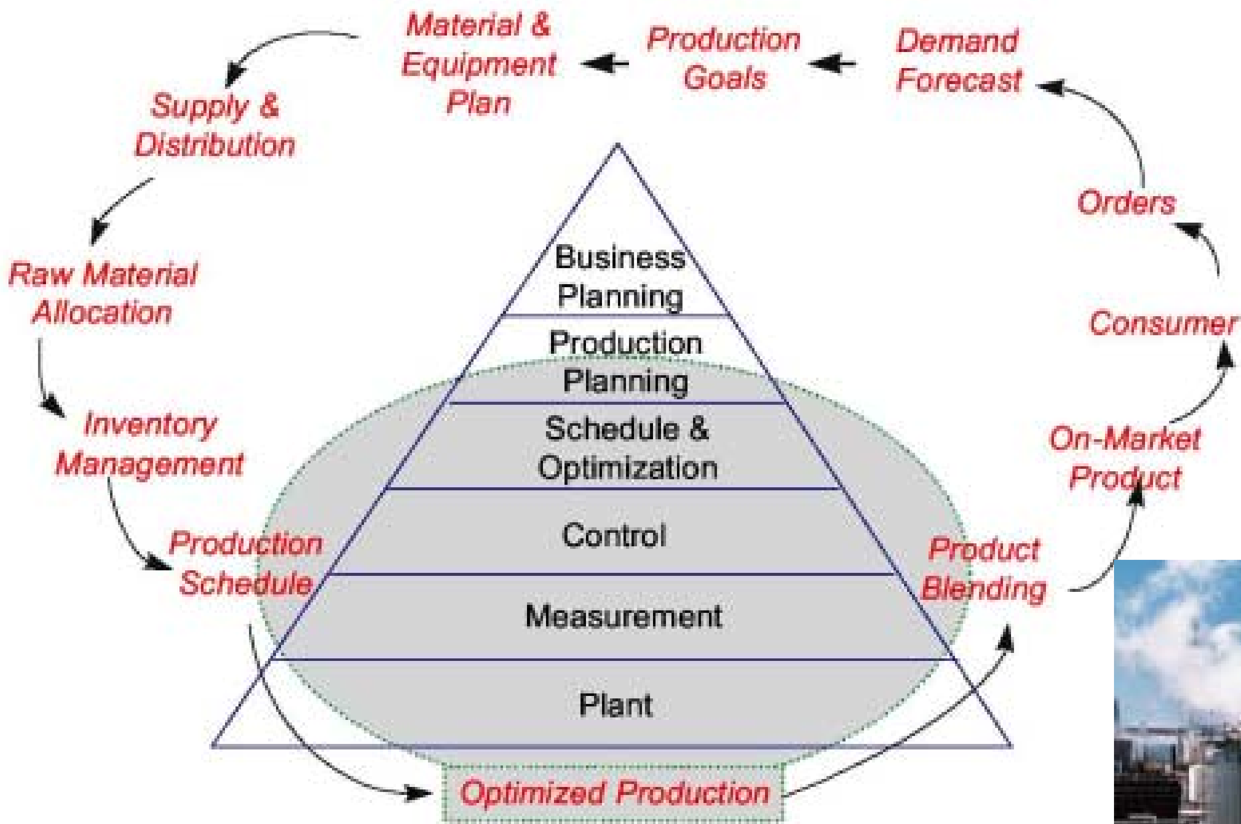
- Need to state problem such that a solution can be computed quickly, efficiently, reliably

- Least squares - linear quadratic problems
  - analytical closed form, matrix multiplication and inversion
- Linear Programming
  - simplex method
- Quadratic Programming
  - interior point
- Convex optimization: includes LP, QP, and more

Optimization in Process Plants



Optimization in Process Plants



Linear programming

- LP Problem:

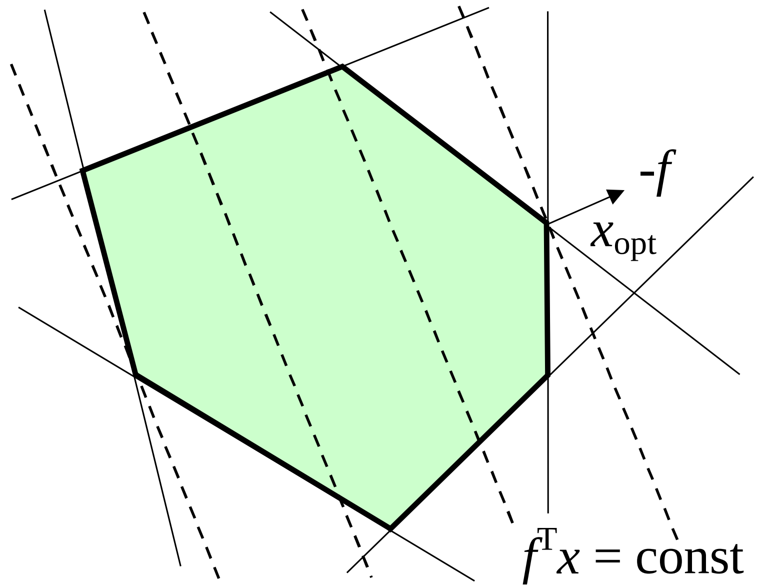
$$Ax \le b$$
$$Gx = h$$
$$J = f^T x \to \min$$

$$x \le y \quad \Leftrightarrow \quad \begin{bmatrix} x_1 \le y_1 \\ \vdots \\ x_n \le y_n \end{bmatrix}$$

- Might be infeasible! … no solution satisfies all constraints
- Matlab Optimization Toolbox: **`LINPROG`**

Linear programming

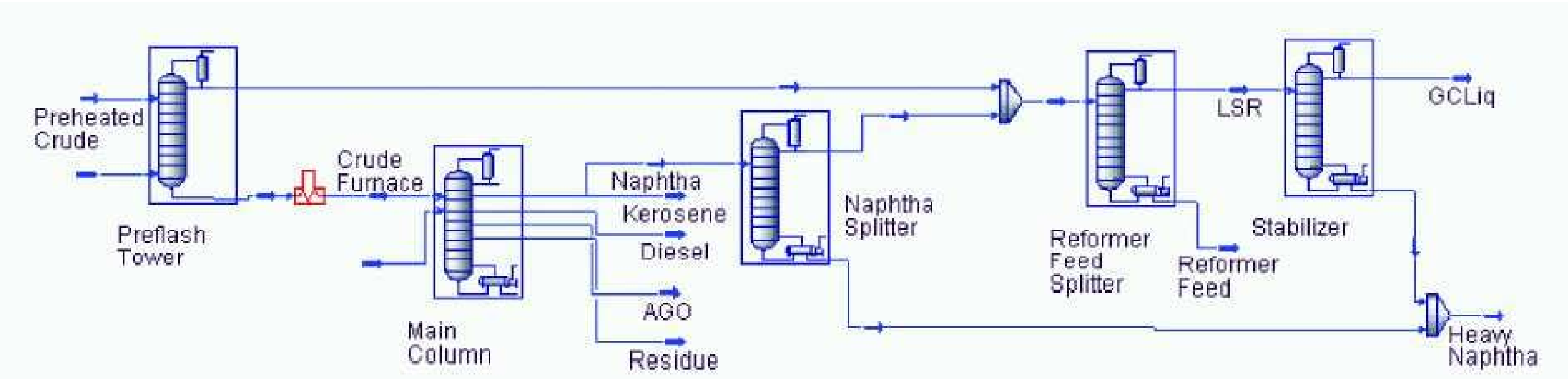
$$Ax \le b$$
$$Gx = h$$
$$J = f^T x \to \min$$



- Simplex method in a nutshell:
  - check the vertices for value of *J*, select optimal
  - issue: exponential growth of number of vertices with the problem size
  - Need to do 10000 variables and 500000 inequalities.
- Modern interior point methods are radically faster
  - no need to understand, standard solvers are available

Refinery Optimization

- Crude supply chain - multiple oil sources
- Distillation - separating fractions
- Blending - ready products, given octane ratings
- Objective function - profit
- LP works ideally:
  - linear equalities and inequalities, single linear objective function



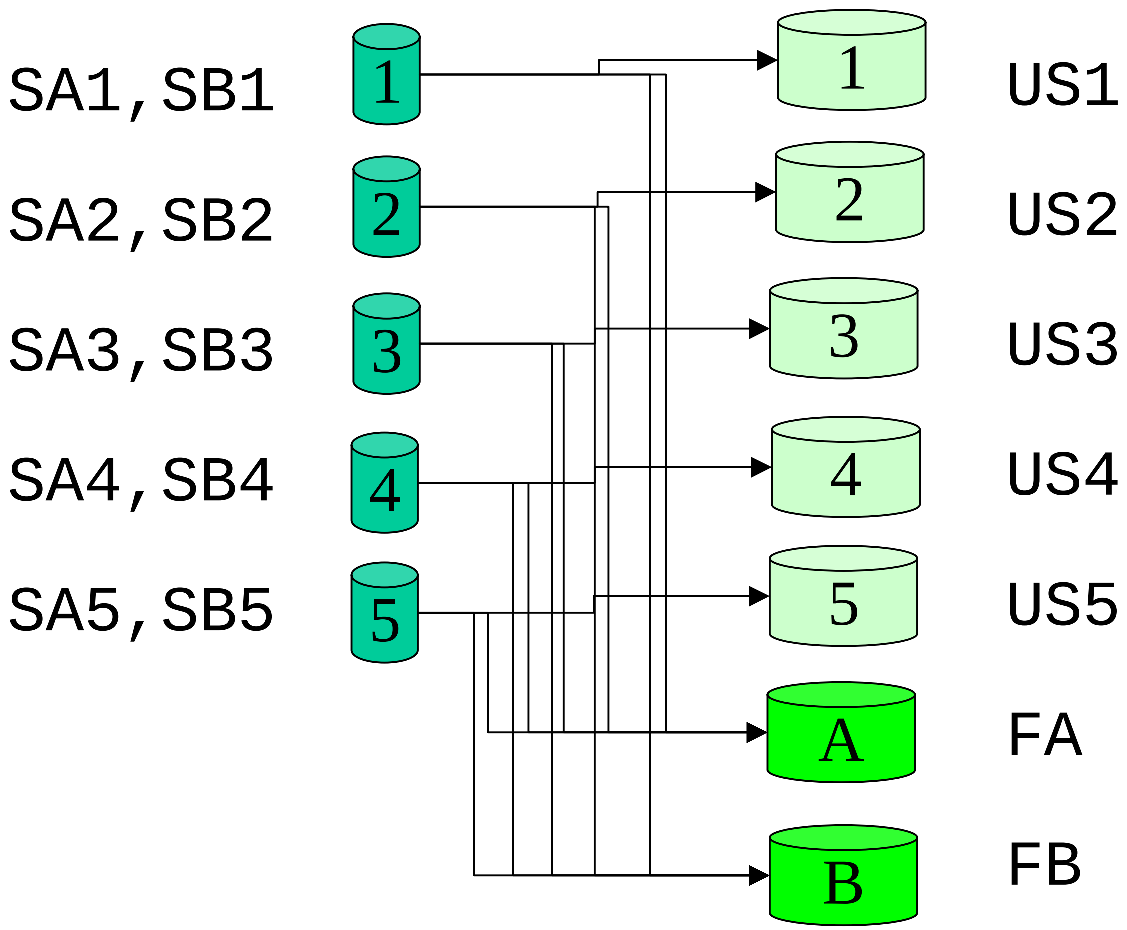
Blending Example

- A Blending Problem: A refinery produces two grades of fuel, A and B, which are made by blending five raw stocks of differing octane rating, cost and availability

Gasoline	Octane Rating	Price $/B
A	93	37.5
B	85	28.5

Stock	Octane Rating	Price $/B	Availability
1	70	9.0	2000
2	80	12.5	4000
3	85	12.5	4000
4	90	27.5	5000
5	99	27.5	3000

Blending Example



Blending Example

- LP problem formulation:

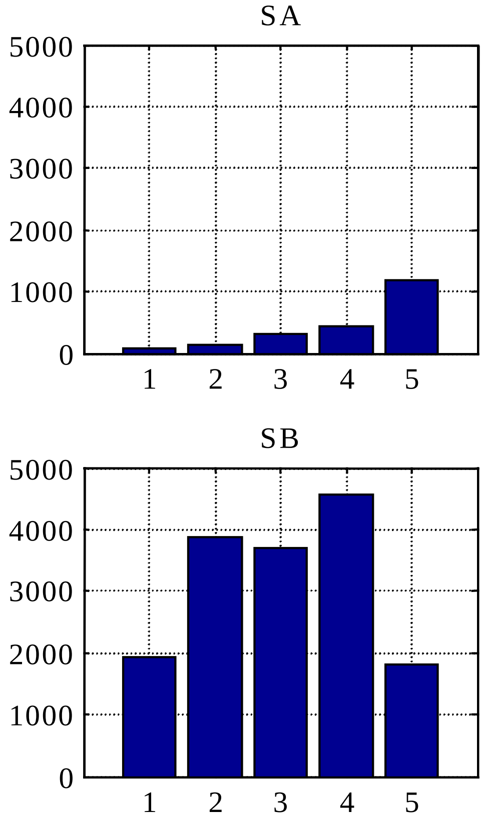
```
J = 9US1 + 12.5US2 + 12.5US3 + 27.5US4 + 27.5US5 + 37.5FA + 28.5FB -> MAX

[Stock Availability]
S1A                +S1B                +US1                = 2000
    S2A            +    S2B            +   US2             = 4000
        S3A        +        S3B        +      US3          = 4000
            S4A    +           S4B     +         US4       = 5000
                S5A+              S5B  +            US5    = 3000

[Fuel Quantity]
S1A+S2A+S3A+S4A+S5A                                       = FA
                    S1B+S2B+S4B+S5B                       = FB

[Fuel Quality]
70S1A + 80S2A + 85S3A + 90S4A + 99S5A                     ≥ 93FA [Quality A]
70S1B + 80S2B + 85S3B + 90S4B + 99S5B                     ≥ 85FB [Quality B]

[Nonnegativity]
S1A,S2A,S3A,S4A,S5A,S1B,S2B,S4B,S5B,US1,US2,US3,US4,US5,FA,FB ≥ 0
```

Matlab code for the example

```
%    OctRt  Price $/B
Gas = [93       37.5;
       85       28.5];
%Stock  OctRt    Price $/B    Availability
Stock = [70        12.5          2000;
         80        12.5          4000;
         85        12.5          4000;
         90        27.5          5000;
         99        27.5          3000];
% Revenue
f = [zeros(10,1); Stock(:,3);  Gas(:,2)];
% Equality constraint
G = [eye(5,5)    eye(5,5)    eye(5,5)    zeros(5,2);
     ones(1,5)   zeros(1,5)  zeros(1,5)  -1    0;
     zeros(1,5)  ones(1,5)   zeros(1,5)   0   -1];
h = [Stock(:,3); zeros(2,1)];
% Inequality (fuel quality) constraints
A = [-[Stock(:,1)' zeros(1,5) zeros(1,5);
       zeros(1,5)  Stock(:,1)' zeros(1,5)] diag(Gas(:,1))];
b = zeros(2,1);
% X=LINPROG(f,A,b,Aeq,beq,LB,UB)
x = linprog(-f,A,b,G,h,zeros(size(f)),[]);
Revenue = f'*x
```

Blending Example - Results

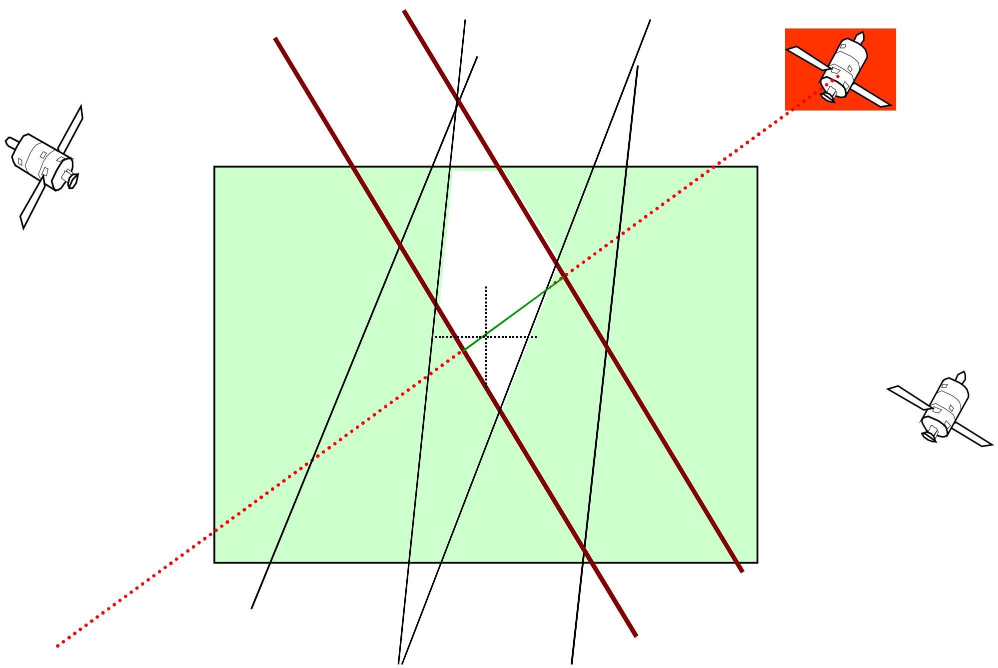
- Blending distribution:



Produced Fuel:		Total Revenue:
A	2125	$532,125
B	15875	

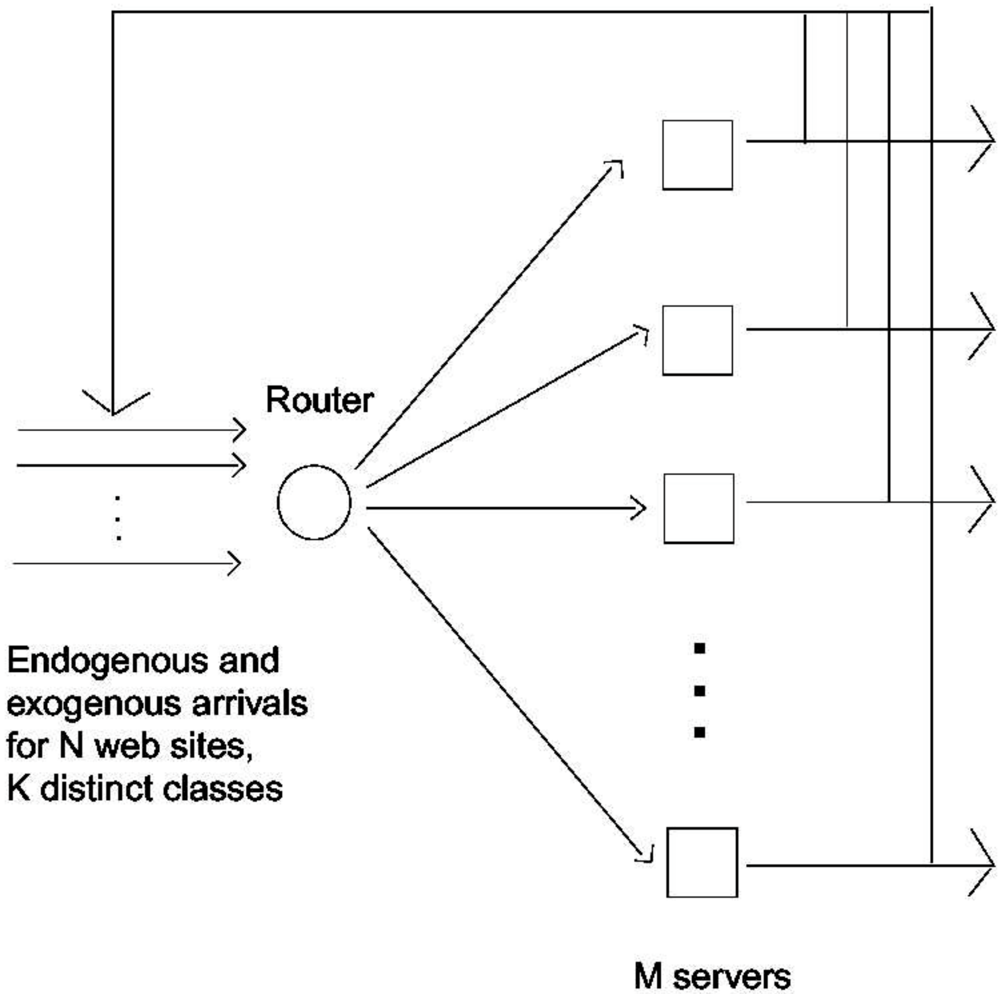
GPS

- Determining coordinates by comparing distances to several satellites with known positions
- See E62 website:
  `http://www.stanford.edu/class/engr62e/handouts/GPSandLP.ppt`

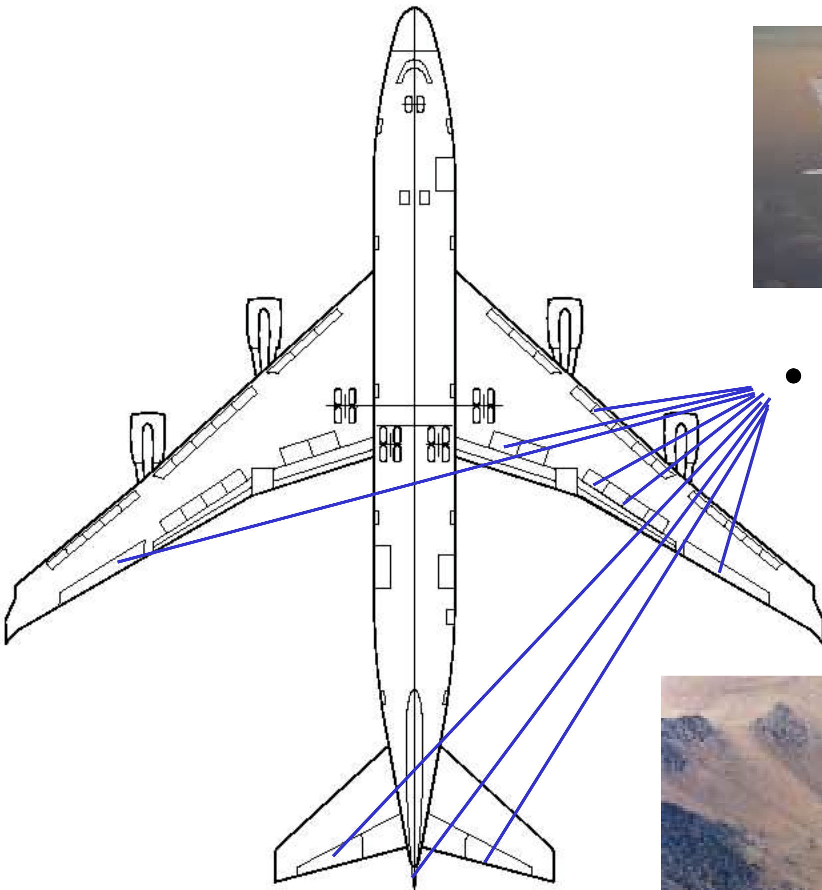


Computing Resource Allocation

- Web Server Farm
- LP formulation for the optimal load distribution



Aircraft actuator allocation



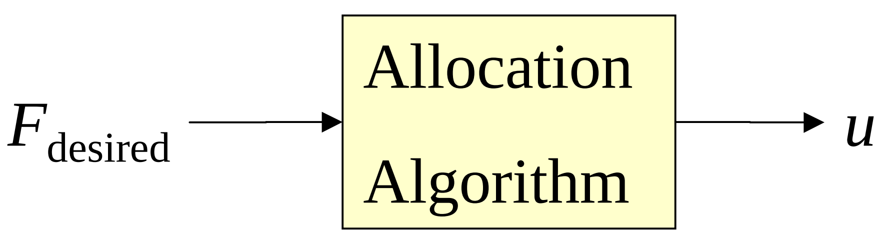
- Mutiple flight control surfaces

Aircraft actuator allocation

- Mutiple flight control surfaces: ailerons, elevons, canard foreplanes, trailing and leading edge flaps, airbrakes, etc

$$\begin{bmatrix} M_{roll} \\ M_{pitch} \\ M_{yaw} \end{bmatrix} = B(\alpha, \varphi, V)u$$

$$F = Bu$$



Actuator allocation

- Simplest approach - least squares

$$u = B^{\dagger} F$$

$$B^{\dagger} = \left(B^T B\right)^{-1} B^T \quad \text{solves} \quad Bu = F, \quad \|u\|_2^2 \to \min$$

- LP optimization approach

$$Bu = F, \quad \|w^T u\|_1 \to \min$$

$$\|w^T u\|_1 = \sum w_k \cdot |u_k|, \qquad w_k \geq 0$$

LP

$$w^T u^+ + w^T u^- \to \min$$

$$u^+ \geq 0$$

$$u^- \geq 0$$

$$Bu^+ - Bu^- = F$$

Solve the LP, get $u = u^+ - u^-$

Actuator allocation

- Need to handle actuator constrains (v - scale factor)

$$\left\| w^T u \right\|_1 - v \rightarrow \min \qquad u^l \le u \le u^u$$

$$Bu = vF \qquad 0 \le v \le 1$$

- LP can be extended to include actuator constrains

$$w^T u^+ + w^T u^- - v \rightarrow \min$$
$$Bu^+ - Bu^- - vF = 0$$
$$u^l \le u^+ \le u^u$$
$$u^l \le -u^- \le u^u$$
$$0 \le v \le 1$$

$$f^T = \begin{bmatrix} w^T & w^T & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} I & 0 & 0 \\ -I & 0 & 0 \\ 0 & -I & 0 \\ 0 & I & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}, b = \begin{bmatrix} u^u \\ -u^l \\ u^u \\ -u^l \\ 1 \\ 0 \end{bmatrix}, \quad x = \begin{bmatrix} u^+ \\ u^- \\ v \end{bmatrix}$$

$$G = \begin{bmatrix} B & -B & -F \end{bmatrix}, h = 0$$

$$Ax \le b$$
$$Gx = h$$
$$f^T x \rightarrow \min$$

Actuator allocation example

- Problem:

$$\left\| w^T u \right\|_1 - v \rightarrow \min$$

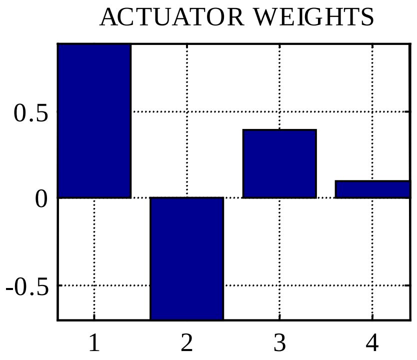
$$Bu = vF$$

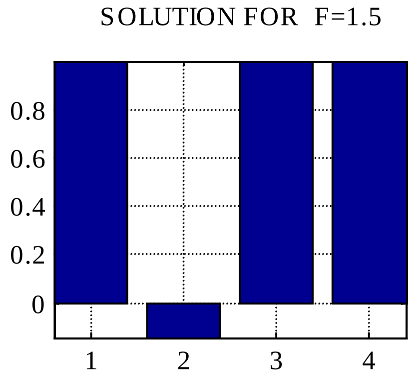
$$B = \begin{bmatrix} 0.9 & -0.7 & 0.4 & 0.1 \end{bmatrix}$$

$$w = \begin{bmatrix} 0.1 & 0.1 & 0.02 & 0.001 \end{bmatrix}$$

$$-1 \leq u \leq 1$$

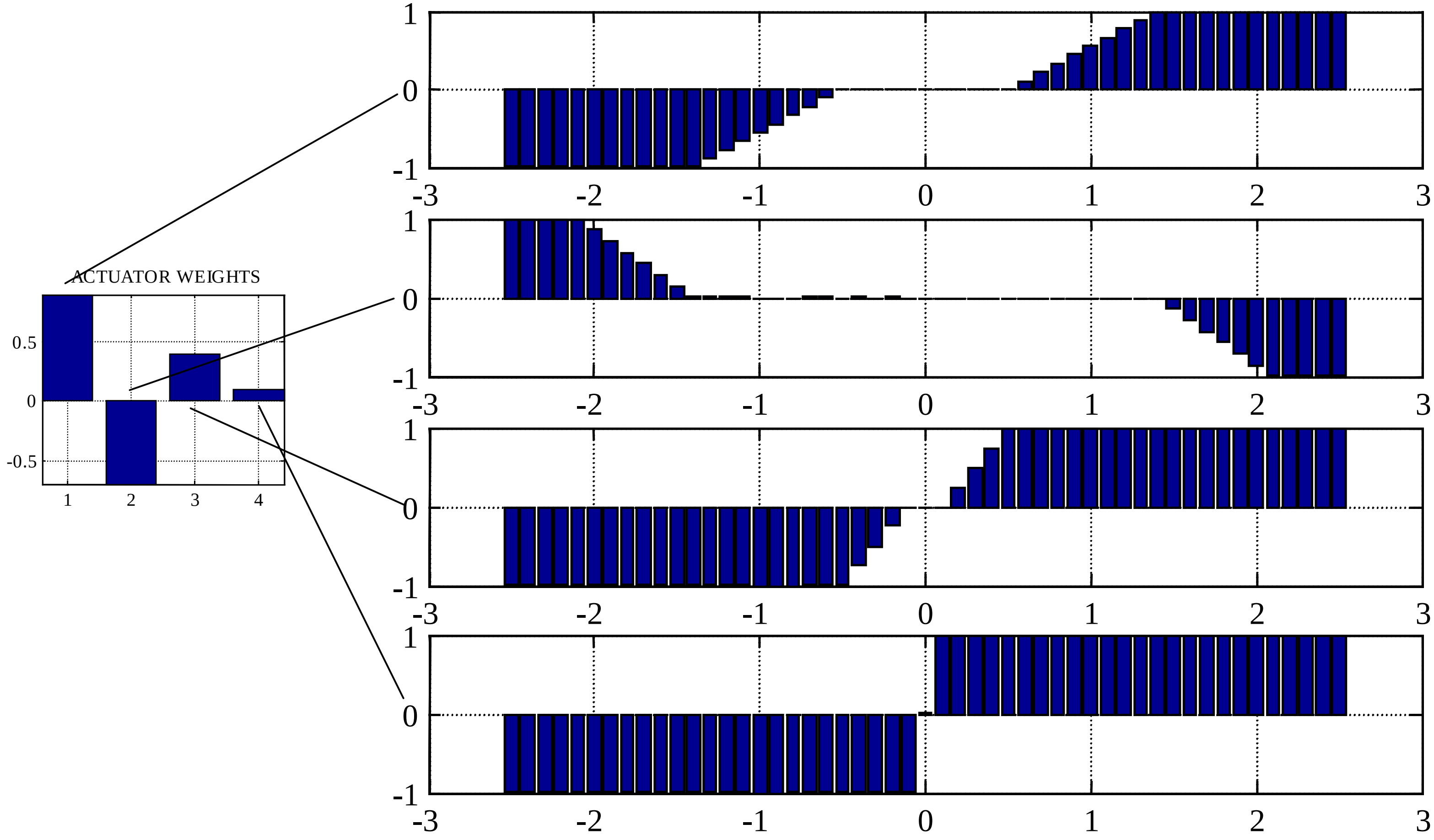
- LP problem solution for $F = 1.5$





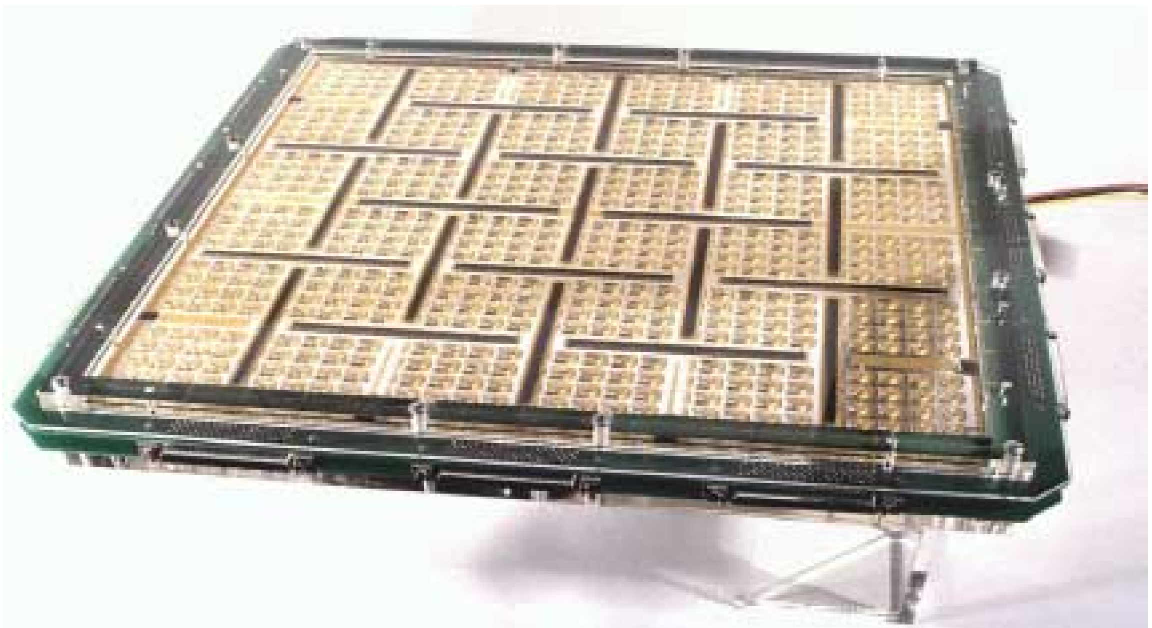
Actuator allocation example

- LP problem solution for *F* from -2.5 to 2.5



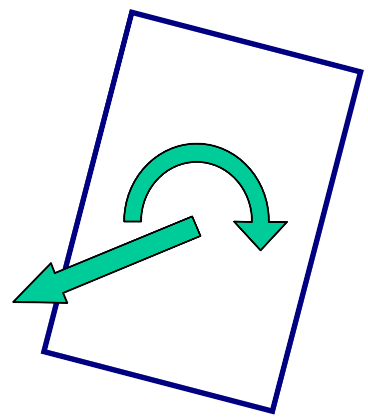
Extreme actuator allocation

- (Xerox) PARC jet array table
- Jets must be allocated to achieve commanded total force and torque acting on a paper sheet



$$\mathbf{F} = \sum \vec{f}_k$$

$$\mathbf{T} = \sum \vec{f}_k \times \vec{r}_k$$



Actuator allocation

- Least squares + actuator constraint

$$Bu = F,$$

$$\|u\|^2 \rightarrow \min$$

$$u^l \le u \le u^u$$

- This is a QP optimization problem

Quadratic Programming

- QP Problem:

$$Ax \le b$$

$$Gx = h$$

$$J = \frac{1}{2}x^T Hx + f^T x \to \min$$

- Matlab Optimization Toolbox: **`QUADPROG`**
- Same feasibility issues as for LP
- Fast solvers available
- More in the next Lecture...