Lecture 2 - Modeling and Simulation

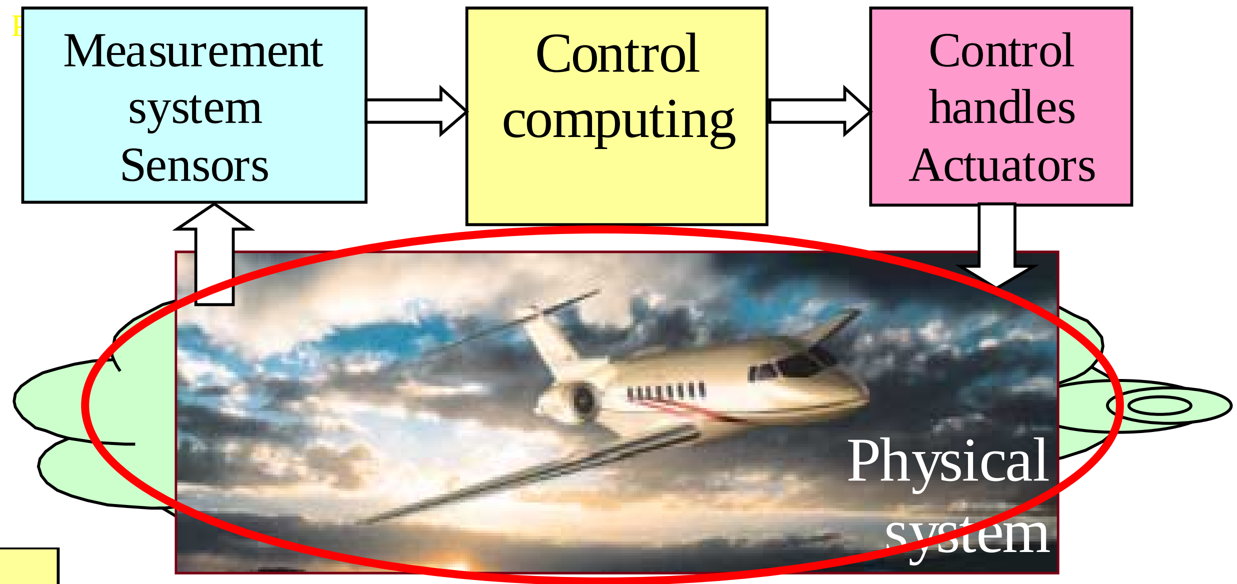
- Model types: ODE, PDE, State Machines, Hybrid
- Modeling approaches:
  - physics based (white box)
  - input-output models (black box)
- Linear systems
- Simulation
- Modeling uncertainty

Goals

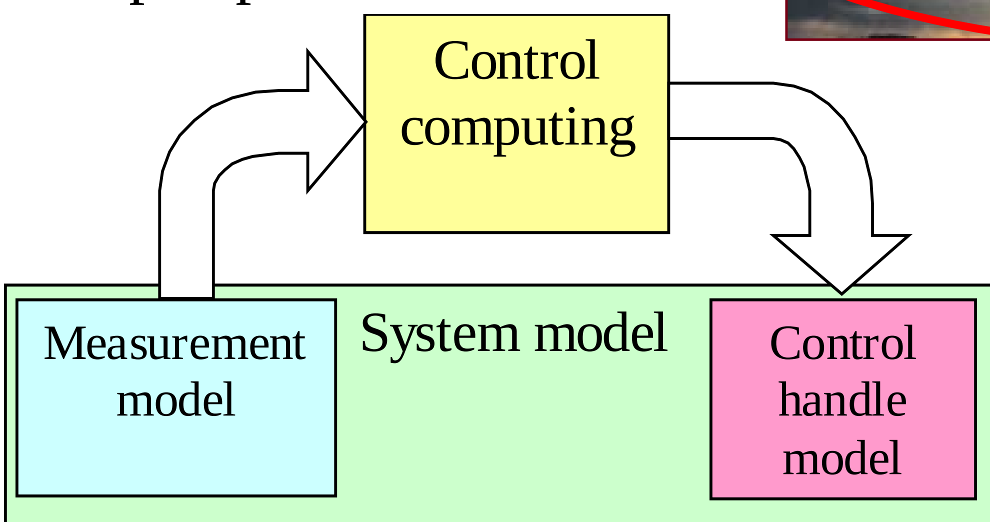
- Review dynamical modeling approaches used for control analysis and simulation
- Most of the material us assumed to be known
- Target audience
  - people specializing in controls - practical

Modeling in Control Engineering

- Control in a system perspective



- Control analysis perspective



Models

- Model is a mathematical representations of a system
  - Models allow simulating and analyzing the system
  - Models are never exact
- Modeling depends on your goal
  - A single system may have many models
  - *Always* understand what is the *purpose* of the model
  - Large ‘libraries’ of standard model templates exist
  - A conceptually new model is a big deal
- Main goals of modeling in control engineering
  - conceptual analysis
  - detailed simulation

Modeling approaches

- Controls analysis uses deterministic models. Randomness and uncertainty are usually not dominant.
- White box models: physics described by ODE and/or PDE
- Dynamics, Newton mechanics

$$\dot{x} = f(x,t)$$

- Space flight: add control inputs u and measured outputs y

$$\dot{x} = f(x,u,t)$$

$$y = g(x,u,t)$$

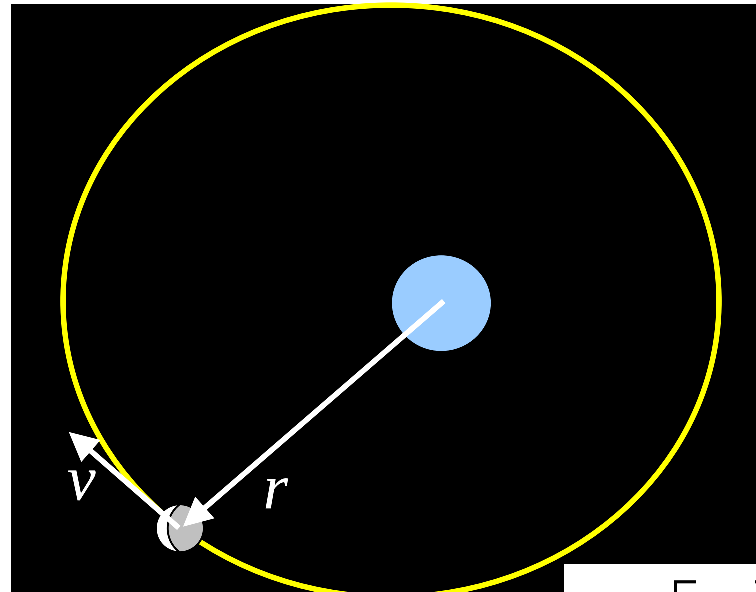
Orbital mechanics example

1643-1736

- Newton's mechanics
  - fundamental laws
  - dynamics

$$\dot{v} = -\gamma m \cdot \frac{r}{|r|^3} + F_{pert}(t)$$

$$\dot{r} = v$$

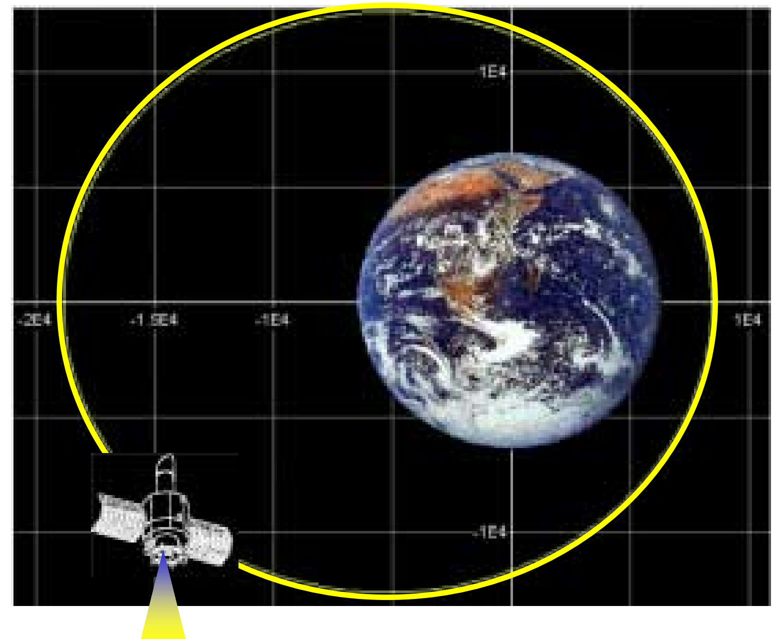


1749-1827

- Laplace
  - computational dynamics (pencil & paper computations)
  - deterministic model-based prediction

$$\dot{x} = f(x,t) \qquad x = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Orbital mechanics example



- Space flight mechanics

Thrust

$$\dot{v} = -\gamma m \cdot \frac{r}{|r|^3} + F_{pert}(t) + u(t)$$

$$\dot{r} = v$$

- Control problems: *u* - ?

observations / measurements

$$y = \begin{bmatrix} \theta(r) \\ \varphi(r) \end{bmatrix}$$

state

$$x = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

model

control

$$\dot{x} = f(x, u, t)$$

$$y = g(x, u, t)$$

Gene expression model

System variables and symbols

O	Total operon concentration
O_F	Free operon concentration
M_F	Free mRNA concentration
E	Total anthranilate synthase concentration
E_A	Active anthranilate synthase concentration
T	Tryptophan concentration
R	Total repressor concentration
R_A	Active repressor concentration
P	mRNA polymerase concentration
ρ	Ribosomal concentration
D	mRNA destroying enzyme concentration

Santillán-Mackey Model Equations

$$\dot{O}_F = \frac{K_r}{K_r + R_A(T)}\{\mu O - k_p P[O_F(t) - O_F(t-\tau_p)e^{-\mu\tau_p}]\} - \mu O_F(t)$$

$$\dot{M}_F = k_p P O_F(t-\tau_m)e^{-\mu\tau_m}[1 - A(T)] - k_\rho \rho[M_F(t) - M_F(t-\tau_\rho)e^{-\mu\tau_\rho}] - (k_d D + \mu)M_F(t)$$

$$\dot{E} = \frac{1}{2}k_\rho \rho M_F(t-\tau_e)e^{-\mu\tau_e} - (\gamma + \mu)E(t)$$

$$\dot{T} = K E_A(E,T) - G(T) + F(T, T_{ext}) - \mu T(t)$$

$$R_A(t) := R\frac{T(t)}{T(t) + K_t}$$

$$A(T) := b(1 - e^{-T(t)/c})$$

$$E_A(E,T) := \frac{K_i^{n_H}}{K_i^{n_H} + T^{n_H}(t)}E(t)$$

$$G(T) := g\frac{T(t)}{T(t) + K_g}$$

$$F(T, T_{ext}) := d\frac{T_{ext}}{e + T_{ext}[1 + T(t)/f]}$$

Sampled Time Models

- Time is often sampled because of the digital computer use
  - computations, numerical integration of continuous-time ODE

$$x(t+d) \approx x(t) + d \cdot f(x,u,t), \qquad t = kd$$

  - digital (sampled time) control system

$$x(t+d) = f(x,u,t)$$

$$y = g(x,u,t)$$

- Time can be sampled because this is how a system works
- Example: bank account balance
  - $x(t)$ - balance in the end of day t
  - $u(t)$ - total of deposits and withdrawals that day
  - $y(t)$ - displayed in a daily statement

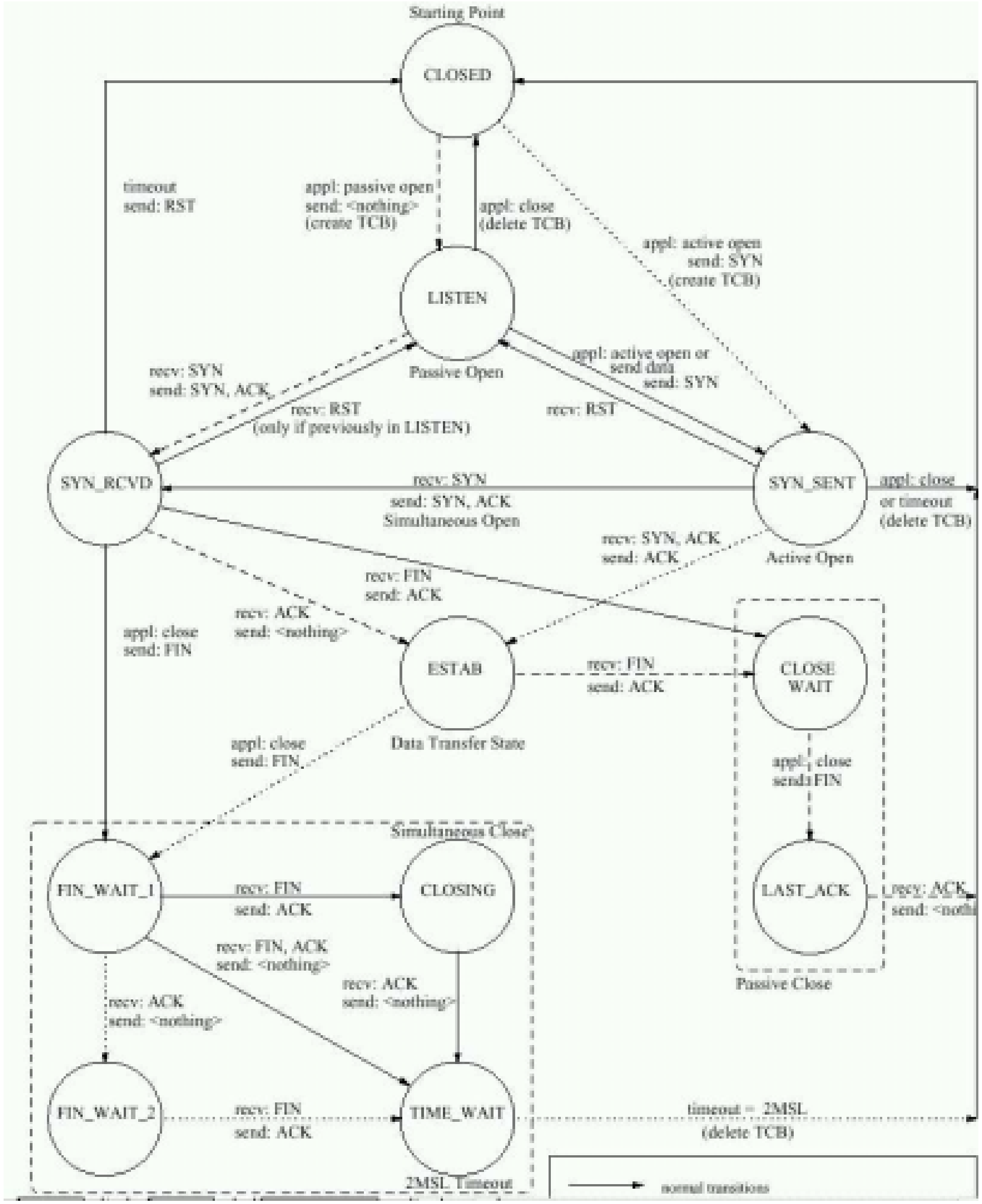
$$x(t+1) = x(t) + u(t)$$

$$y = x$$

- Unit delay operator z^{-1}: $z^{-1}x(t) = x(t-1)$

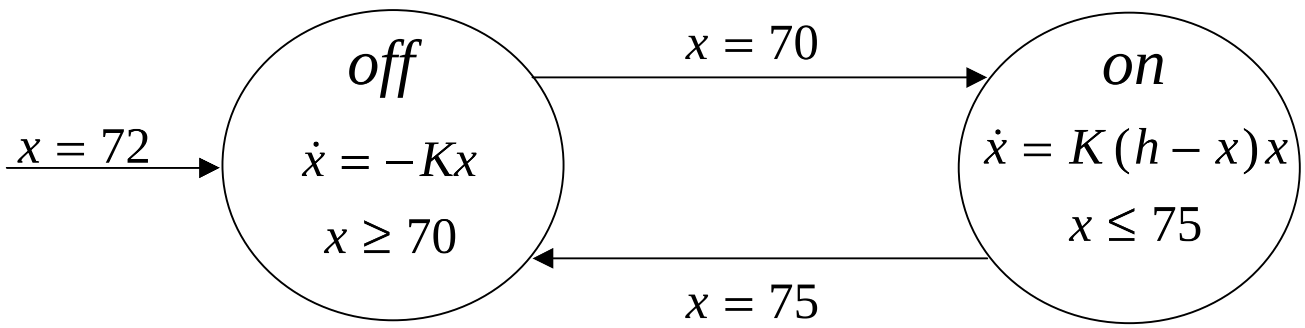
Finite state machines

- TCP/IP State Machine



Hybrid systems

- Combination of continuous-time dynamics and a state machine
- Thermostat example
- Tools are not fully established yet



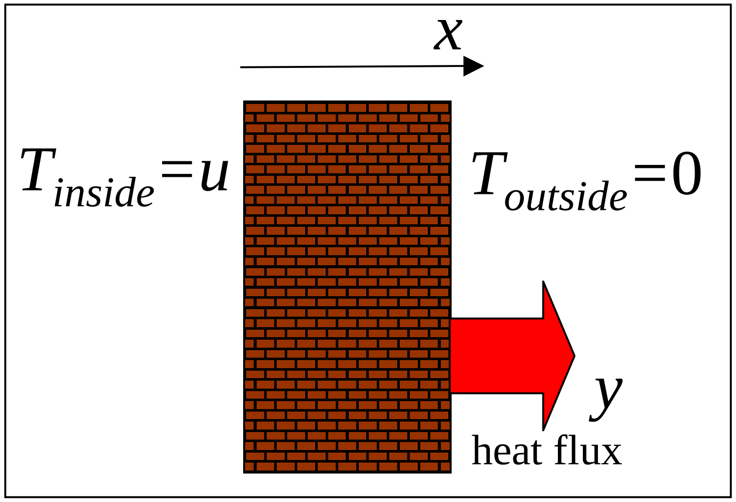
PDE models

- Include functions of spatial variables
  - electromagnetic fields
  - mass and heat transfer
  - fluid dynamics
  - structural deformations
- Example: sideways heat equation

$$\frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}$$

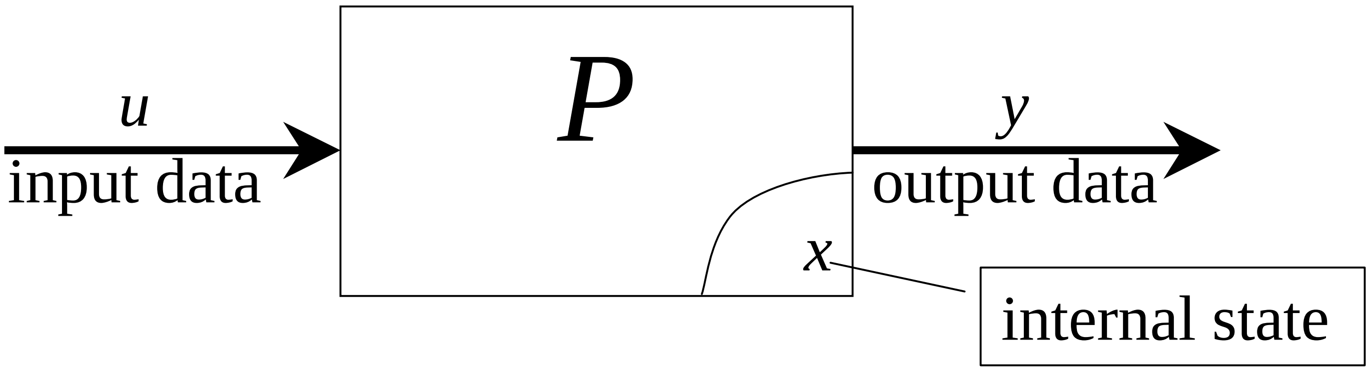
$$T(0) = u; \qquad T(1) = 0$$

$$y = \left.\frac{\partial T}{\partial x}\right|_{x=1}$$



Black-box models

- Black-box models - describe *P* as an operator



  - AA, ME, Physics - state space, ODE and PDE
  - EE - black-box,
  - ChE - use anything
  - CS - state machines, probablistic models, neural networks

Linear Systems

- Impulse response
- FIR model
- IIR model
- State space model
- Frequency domain
- Transfer functions
- Sampled vs. continuous time
- Linearization

Linear System (black-box)

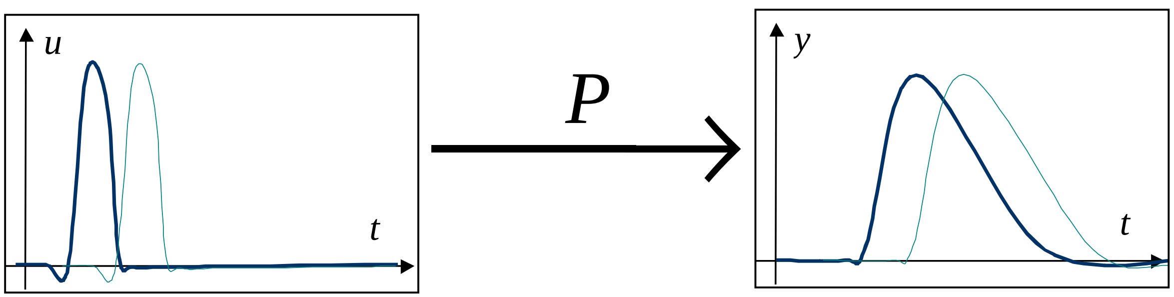
- Linearity

$$u_1(\cdot) \xrightarrow{P} y_1(\cdot) \qquad u_2(\cdot) \xrightarrow{P} y_2(\cdot)$$

$$au_1(\cdot) + bu_2(\cdot) \xrightarrow{P} ay_1(\cdot) + by_2(\cdot)$$

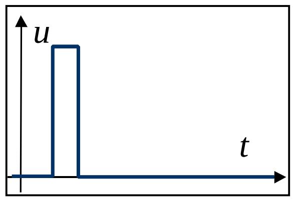
- Linear Time-Invariant systems - LTI

$$u(\cdot - T) \xrightarrow{P} y(\cdot - T)$$

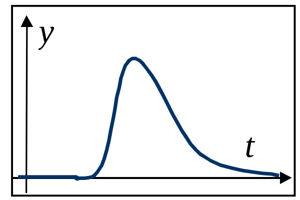


Impulse response

- Response to an input impulse



$$\delta(\cdot) \xrightarrow{P} h(\cdot)$$



- Sampled time: t = 1, 2, ...
- Control history = linear combination of the impulses $\Rightarrow$ system response = linear combination of the impulse responses

$$u(t) = \sum_{k=0}^{\infty} \delta(t-k) u(k)$$

$$y(t) = \sum_{k=0}^{\infty} h(t-k) u(k) = (h * u)(t)$$

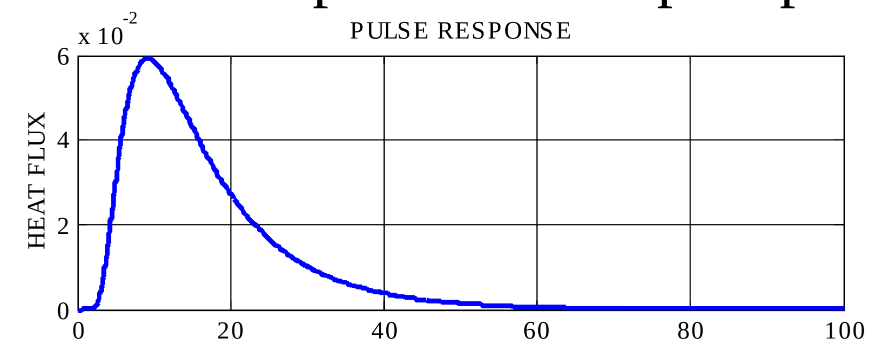
Linear PDE System Example

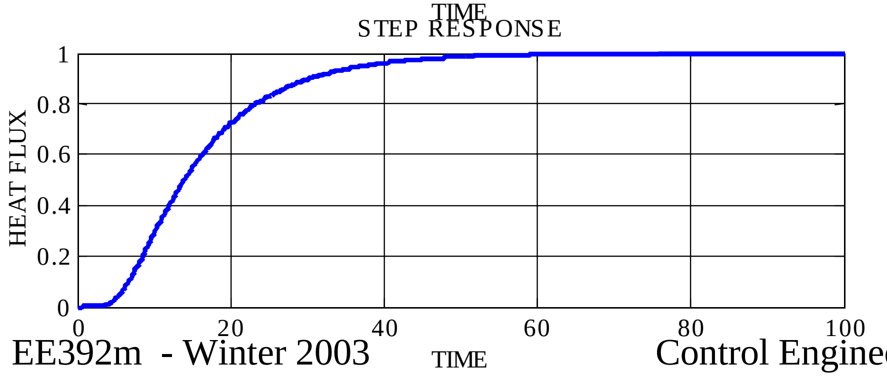
- Heat transfer equation,
  - boundary temperature input *u*
  - heat flux output *y*
- Pulse response and step response

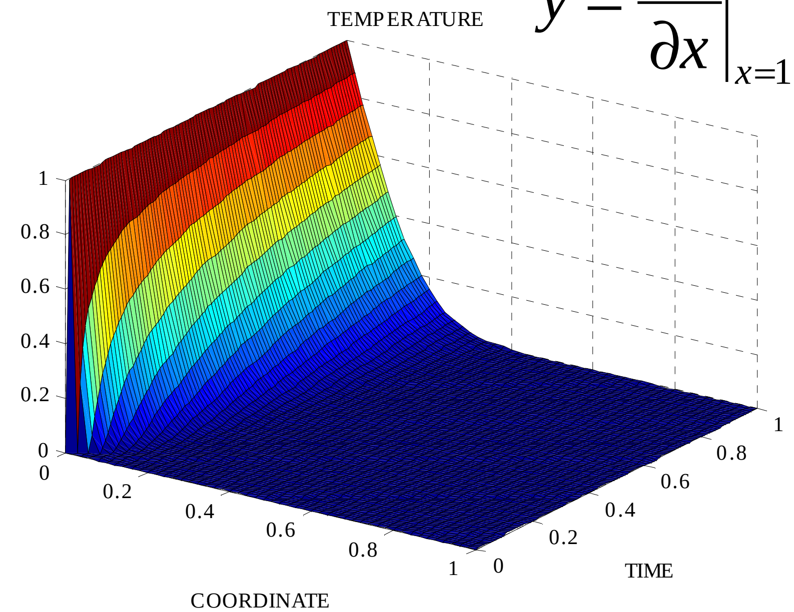
$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}$$

$$u = T(0) \qquad T(1) = 0$$

$$y = \left.\frac{\partial T}{\partial x}\right|_{x=1}$$





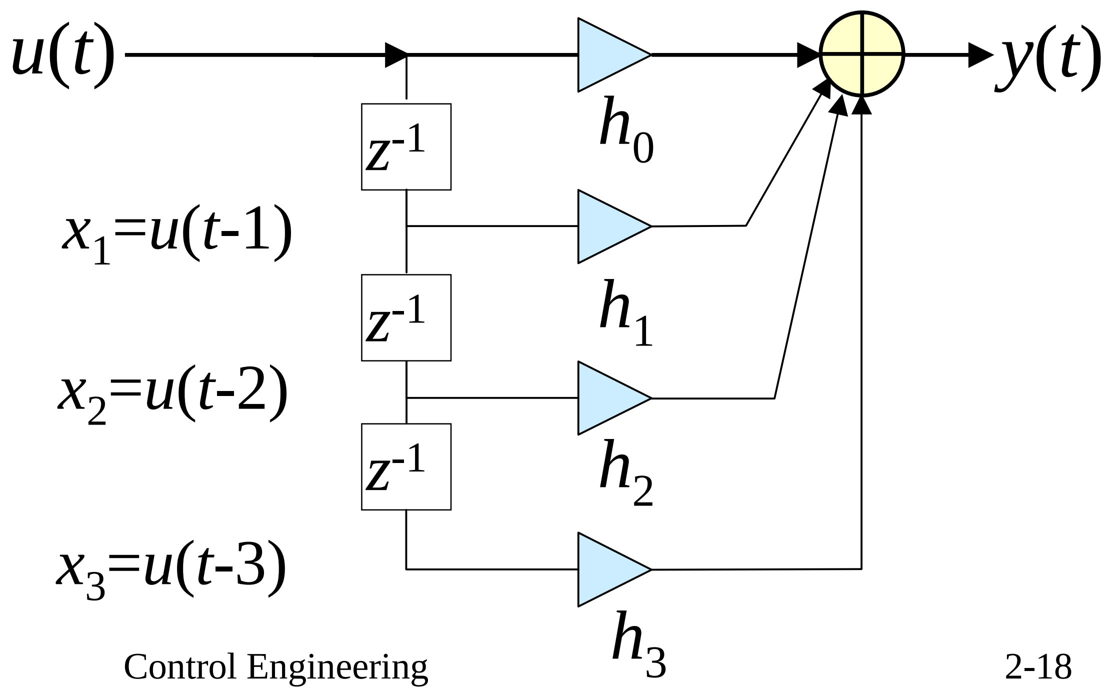


FIR model

$$y(t) = \sum_{k=0}^{N} h_{FIR}(t-k)u(k) = (h_{FIR} * u)(t)$$

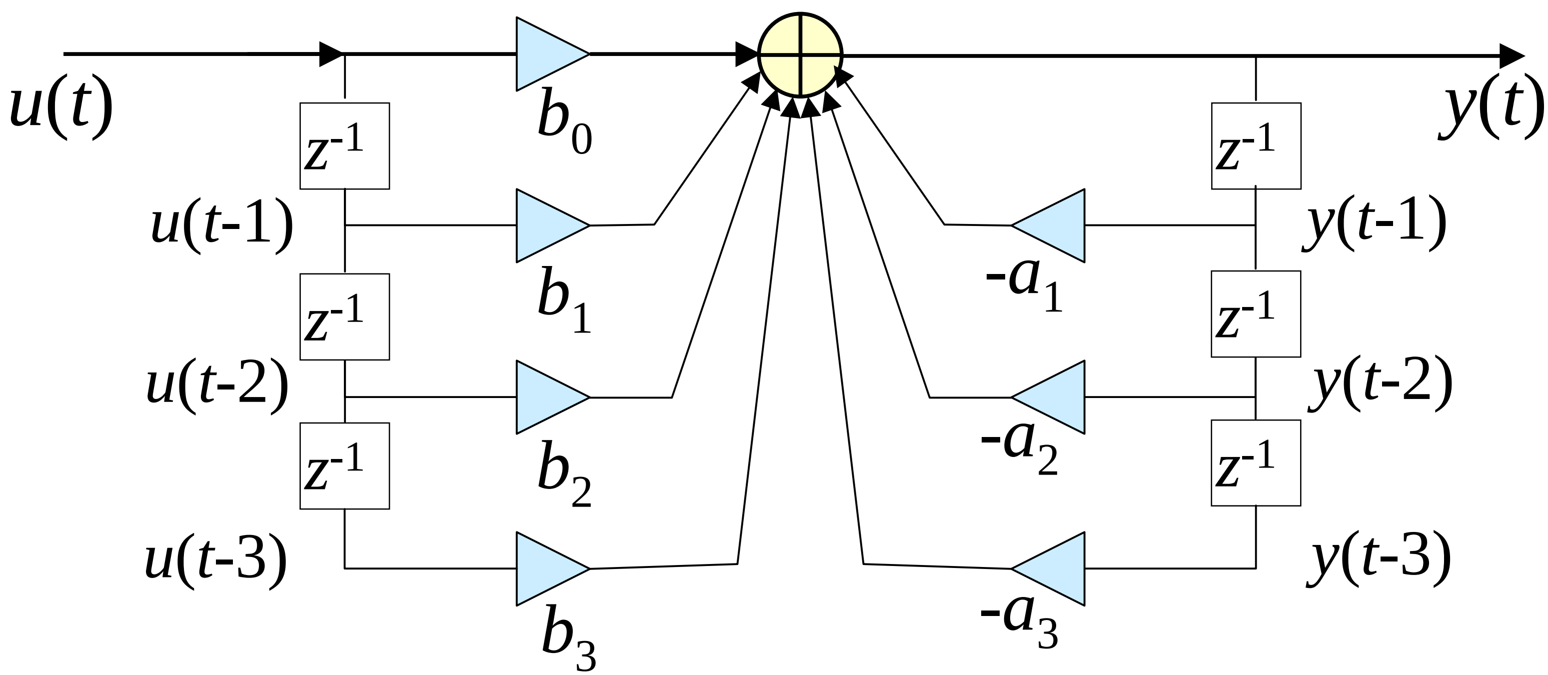
- FIR = Finite Impulse Response
- Cut off the trailing part of the pulse response to obtain FIR
- FIR filter state *x*. Shift register

$$x(t+1) = f(x,u)$$
$$y = g(x,u)$$



IIR model

- IIR model: $y(t) = -\sum_{k=1}^{n_a} a_k y(t-k) + \sum_{k=0}^{n_b} b_k u(t-k)$
- Filter states: $y(t\text{-}1), \ldots, y(t\text{-}n_a), u(t\text{-}1), \ldots, u(t\text{-}n_b)$



IIR model

- Matlab implementation of an IIR model: **`filter`**
- Transfer function realization: unit delay operator z^{-1}

$$y(t) = H(z)u(t)$$

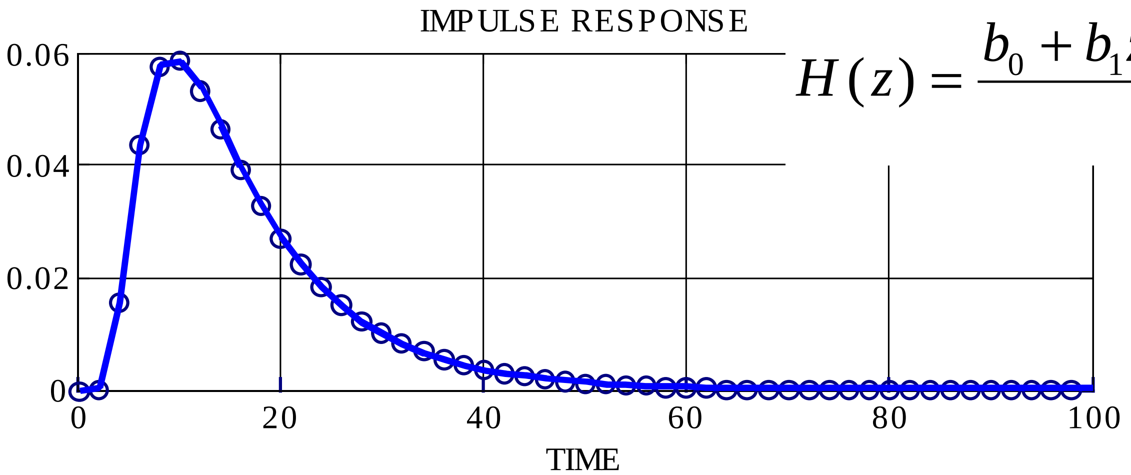
$$H(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + ... + b_N z^{-N}}{1 + a_1 z^{-1} + ... + a_N z^{-N}} = \frac{b_0 z^N + b_1 z^{N-1} + ... + b_N}{z^N + a_1 z^{N-1} + ... + a_N}$$

$$\underbrace{\left(1 + a_1 z^{-1} + ... + a_N z^{-N}\right)}_{A(z)} y(t) = \underbrace{\left(b_0 + b_1 z^{-1} + ... + b_N z^{-N}\right)}_{B(z)} u(t)$$

- FIR model is a special case of an IIR with $A(z) = 1$ (or z^N)

IIR approximation example

- Low order IIR approximation of impulse response: (**`prony`** in Matlab Signal Processing Toolbox)
- Fewer parameters than a FIR model
- Example: sideways heat transfer
  - pulse response $h(t)$
  - approximation with IIR filter $a = [a_1\ a_2]$, $b=[b_0\ b_1\ b_2\ b_3\ b_4]$



$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + b_4 z^{-4}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

Linear state space model

- Generic state space model:

$$x(t+1) = f(x,u,t)$$
$$y = g(x,u,t)$$

- LTI state space model
  - another form of IIR model
  - physics-based linear system model

$$\boxed{\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}}$$

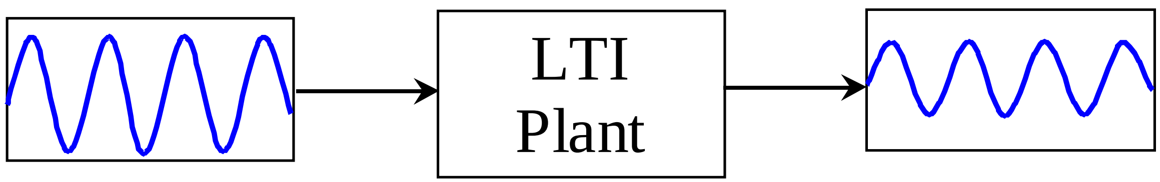
- Transfer function of an LTI model
  - defines an IIR representation

$$y = \left[(Iz - A)^{-1}B + D\right] \cdot u$$
$$H(z) = (Iz - A)^{-1}B + D$$

- Matlab commands for model conversion: **`help ltimodels`**

Frequency domain description

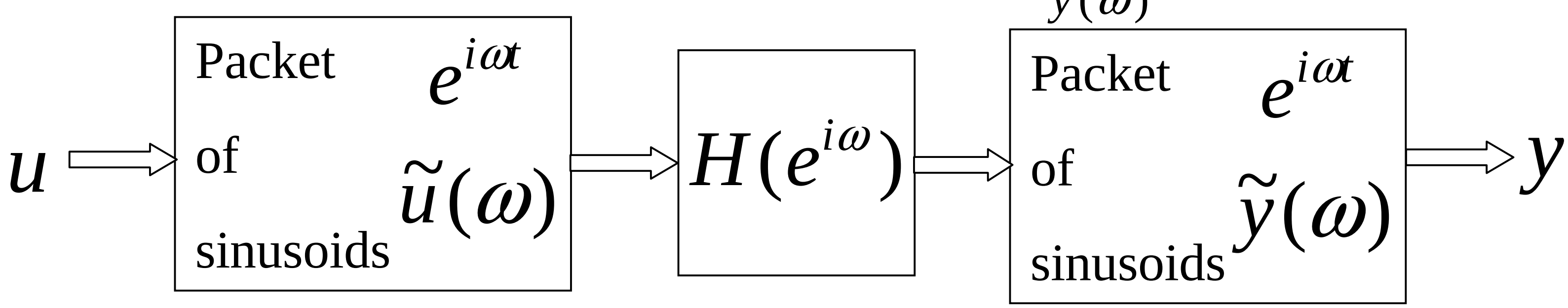
- Sinusoids are eigenfunctions of an LTI system: $y = H(z)u$



$$z^{-1}e^{i\omega t} = e^{i\omega(t-1)} = e^{-i\omega}e^{i\omega t}$$

- Frequency domain analysis

$$u = \int \tilde{u}(\omega)e^{i\omega t}d\omega \Rightarrow y = \int \underbrace{H(e^{i\omega})\tilde{u}(\omega)}_{\tilde{y}(\omega)}e^{i\omega t}d\omega$$



Frequency domain description

- Bode plots:

$$u = e^{i\omega t}$$

$$y = H(e^{i\omega})e^{i\omega t}$$

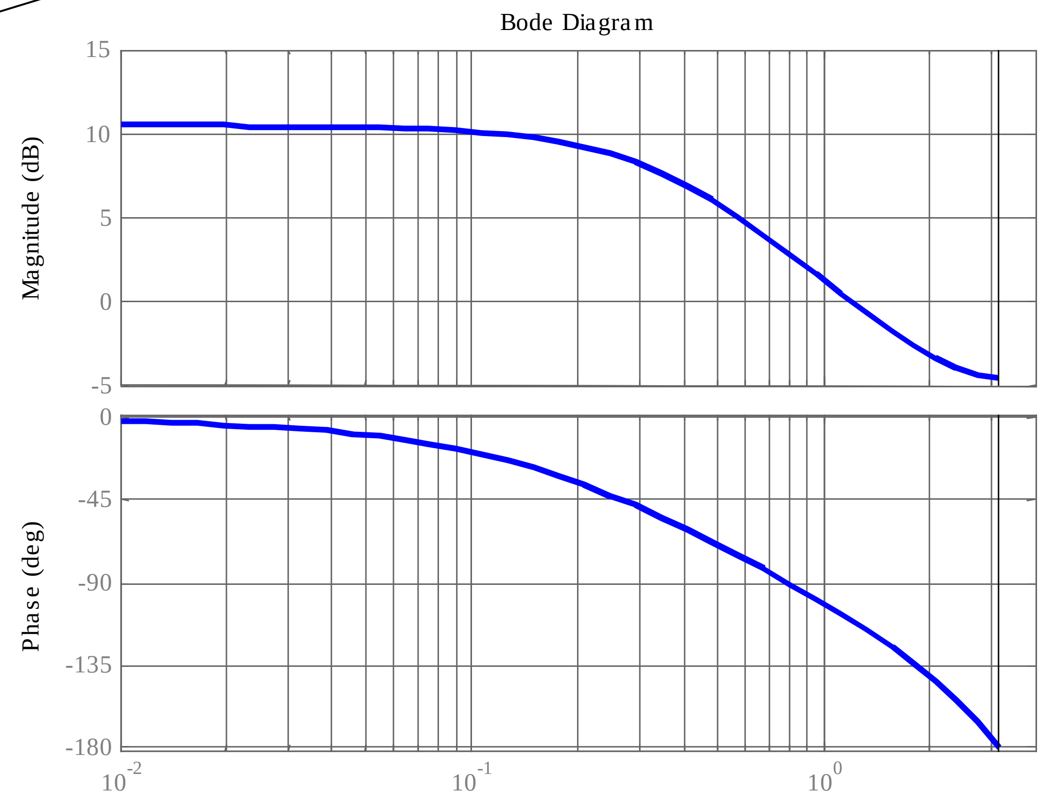
$$M(\omega) = \left| H(e^{i\omega}) \right|$$

$$\varphi(\omega) = \arg H(e^{i\omega})$$

- Example:

$$H(z) = \frac{1}{z - 0.7}$$

- $|H|$ is often measured in dB



Black-box model from data

- Linear black-box model can be determined from the data, e.g., step response data
- This is called model identification
- Lecture 8

z-transform, Laplace transform

- Formal description of the transfer function:
  - function of complex variable z
  - analytical outside the circle $|z| \geq r$
  - for a stable system $r \leq 1$

$$H(z) = \sum_{k=0}^{\infty} h(k) z^{-k}$$

- Laplace transform:
  - function of complex variable s
  - analytical in a half plane $\mathrm{Re}\, s \leq a$
  - for a stable system $a \leq 1$

$$H(s) = \int_{-\infty}^{\infty} h(t) e^{st} dt$$

$$\hat{y}(s) = H(s)\hat{u}(s)$$

Stability analysis

- Transfer function poles tell you everything about stability
- Model-based analysis for a simple feedback example:

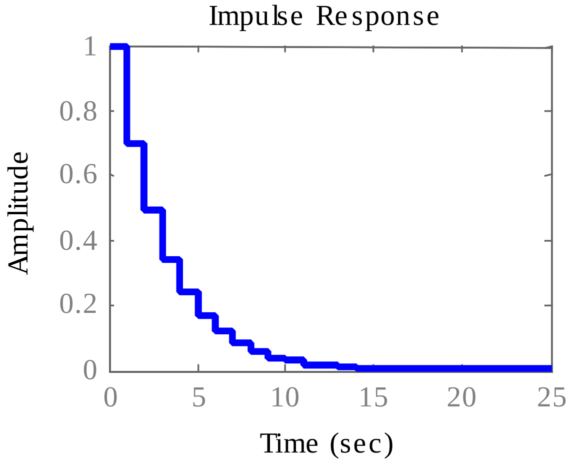
$$y = H(z)u$$
$$u = -K(y - y_d)$$
$$\Longrightarrow \quad y = \frac{H(z)K}{1 + H(z)K} y_d = L(z) y_d$$

- If $H(z)$ is a rational transfer function describing an IIR model
- Then $L(z)$ also is a rational transfer function describing an IIR model

Poles and Zeros <=> System

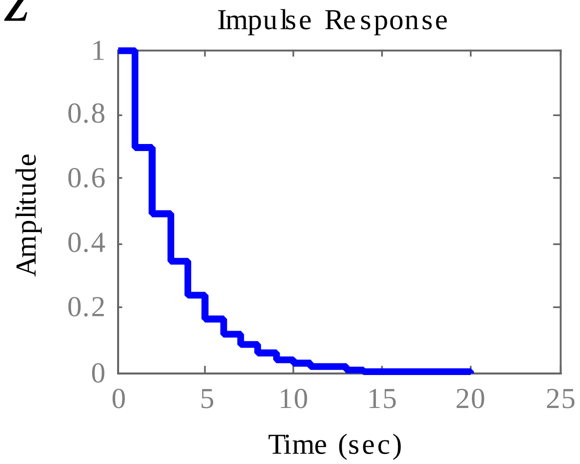
- …**not quite so!**
- Example:

$$y = H(z)u = \frac{z}{z-0.7}$$



- FIR model - truncated IIR

$$y = H_{FIR}(z)u = \frac{z^{19} + 0.7z^{18} + 0.49z^{17} + \ldots + 0.001628\,z + 0.00114}{z^{19}}$$



IIR/FIR example - cont'd

- Feedback control:

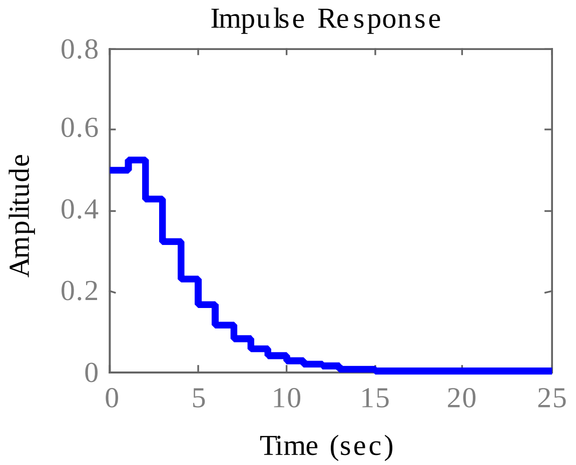
$$y = H(z)u = \frac{z}{z-0.7}$$

$$u = -K(y - y_d) = -(y - y_d)$$

- Closed loop:

$$y = \frac{H(z)}{1+H(z)}u = L(z)u$$

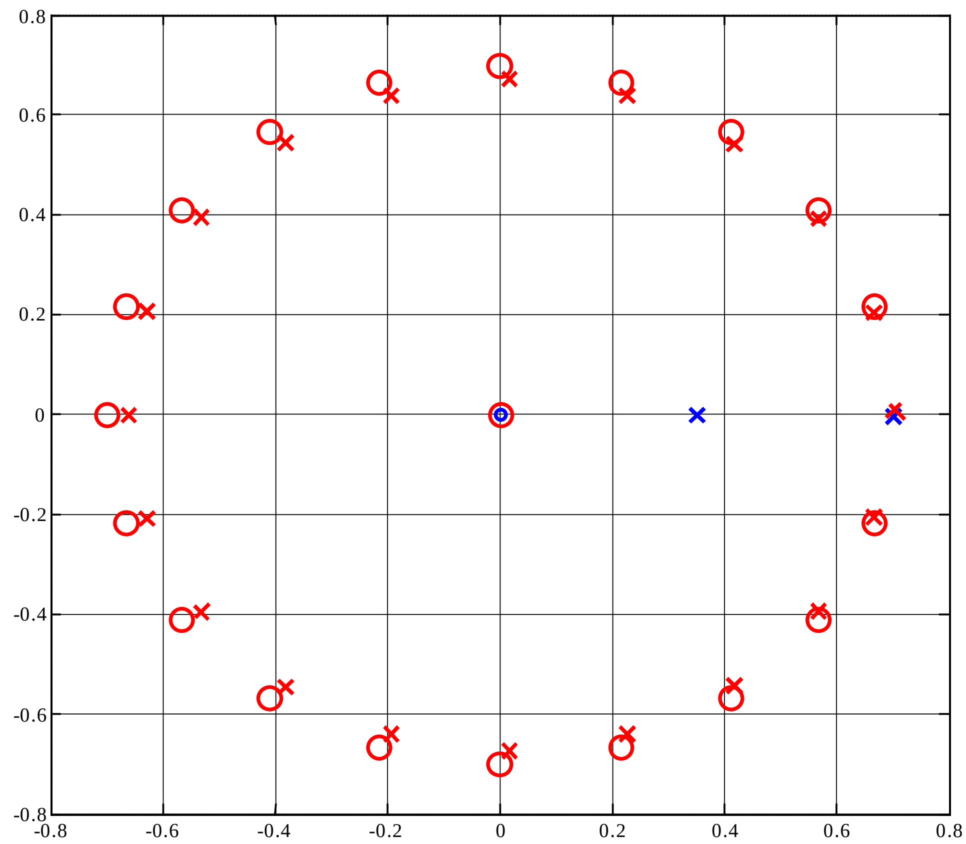
$$y = \frac{H_{FIR}(z)}{1+H_{FIR}(z)}u = L_{FIR}(z)u$$



IIR/FIR example - cont'd
Poles and zeros

- **Blue:** Loop with IIR model poles **x** and zeros **o**

- **Red**: Loop with FIR model poles **x** and zeros **o**



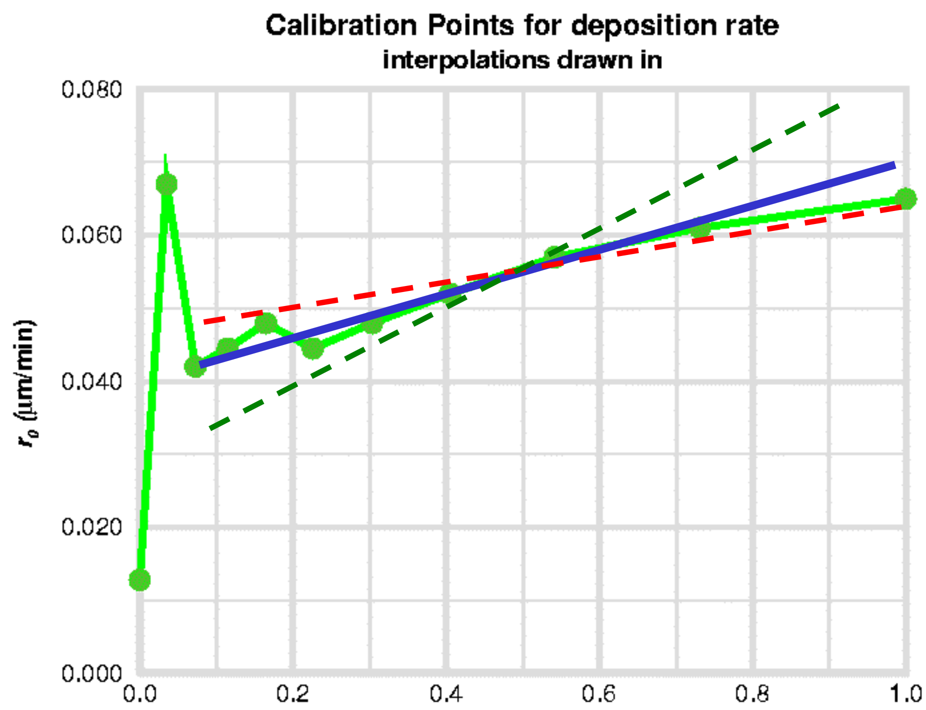
LTI models - summary

- Linear system can be described by impulse response
- Linear system can be described by frequency response = Fourier transform of the impulse response
- FIR, IIR, State-space models can be used to obtain close approximations of a linear system
- A pattern of poles and zeros can be very different for a small change in approximation error.
- Approximation error <=> model uncertainty

Nonlinear map linearization

- Nonlinear - detailed model
- Linear - conceptual design model
- Static map, gain range, sector linearity
- Differentiation, secant method

$$y = f(u) \approx \frac{\Delta f}{\Delta u}(u - u_0)$$



Nonlinear state space model linearization

- Linearize the r.h.s. map

$$\dot{x} = f(x,u) \approx \frac{\Delta f}{\Delta x}\underbrace{(x - x_0)}_{q} + \frac{\Delta f}{\Delta u}\underbrace{(u - u_0)}_{v}$$

$$\dot{q} = Aq + Bv$$

- Secant method

$$\left[\frac{\Delta f}{\Delta x}\right]^j = \frac{f(x + s_j)}{s_j}$$

$$s_j = [0 \quad ... \quad \underbrace{1}_{\# j} \quad ... \quad 0]$$

- Or … capture a response to small step and build an impulse response model

Sampled time vs. continuous time

- Continuous time analysis (Digital implementation of continuous time controller)
  - Tustin's method = trapezoidal rule of integration for $H(s) = \frac{1}{s}$

$$H(s) \rightarrow H_s(z) = H\left(s = \frac{2}{T} \cdot \frac{1 - z^{-1}}{1 + z^{-1}} \right)$$

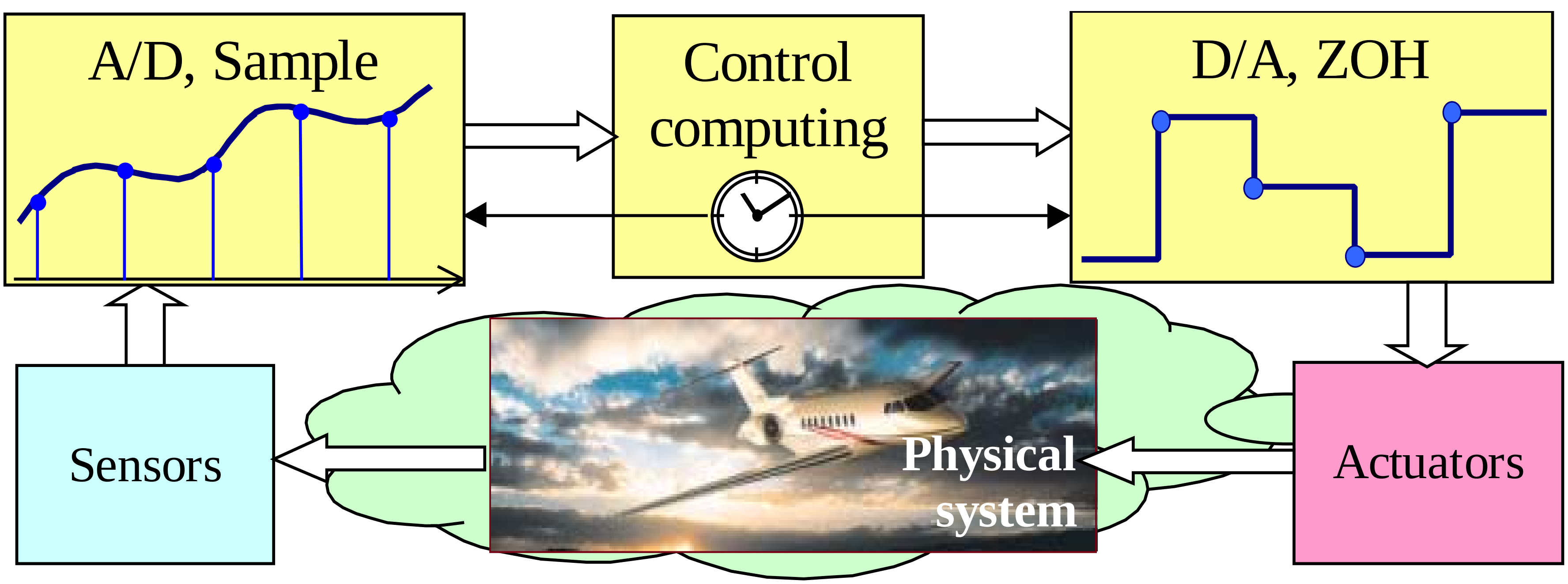
  - Matched Zero Pole: map each zero and a pole in accordance with

$$s = e^{sT}$$

- Sampled time analysis (Sampling of continuous signals and system)

Sampled and continuous time

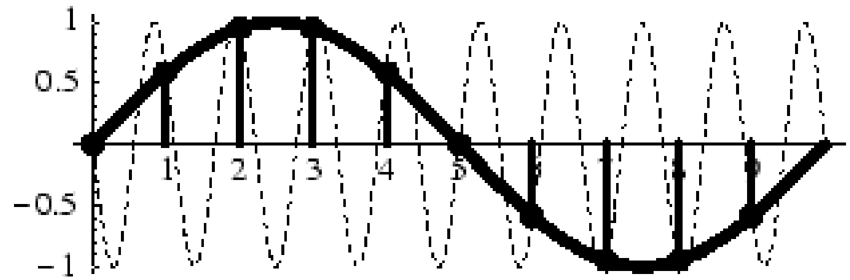
- Sampled and continuous time together
- Continuous time physical system + digital controller
  - ZOH = Zero Order Hold



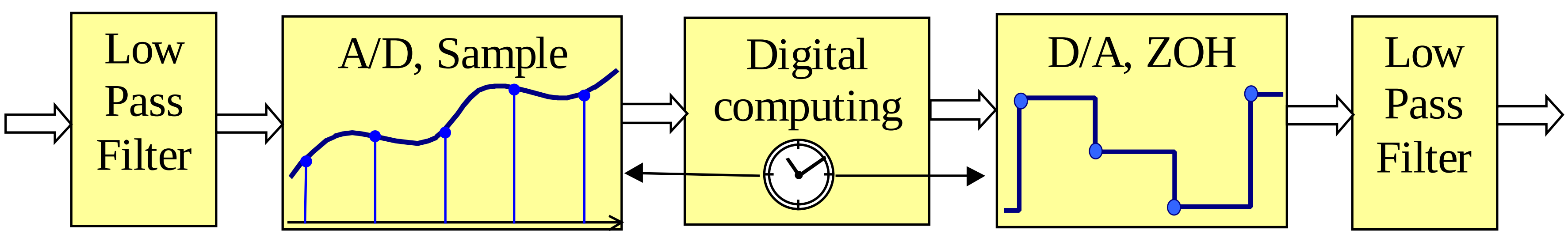
Signal sampling, aliasing

- Nyquist frequency:

  $\omega_N = ½\omega_S$; $\omega_S = 2\pi/T$



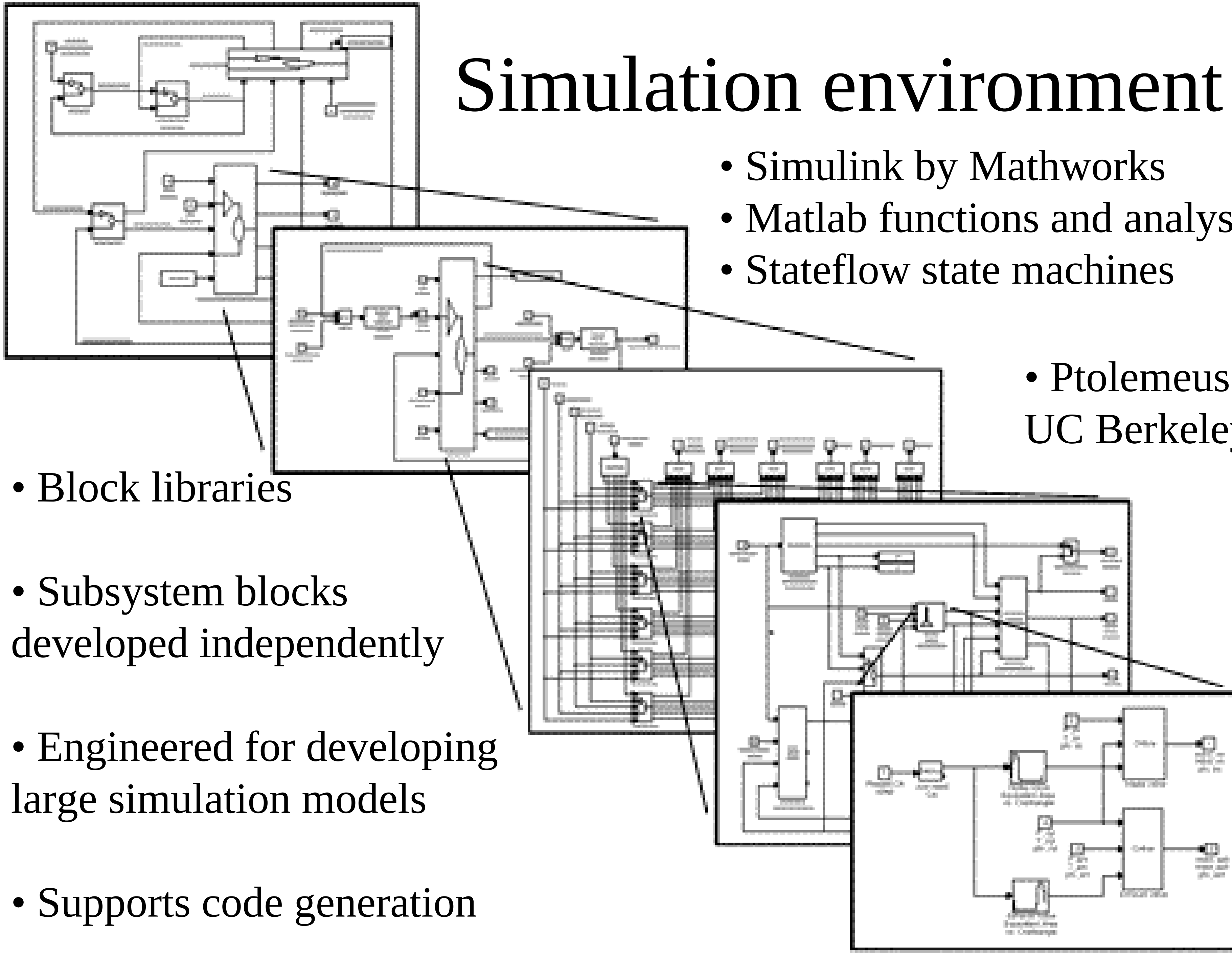
- Frequency folding: $k\omega_S \pm \omega$ map to the same frequency ω
- Sampling Theorem: sampling is OK if there are no frequency components above ω_N
- Practical approach to anti-aliasing: low pass filter (LPF)
- Sampled→continuous: impostoring



Simulation

- ODE solution
  - dynamical model: $\dot{x} = f(x,t)$
  - Euler integration method: $x(t+d) = x(t) + d \cdot f\left(x(t),t\right)$
  - Runge-Kutta: **ode45** in Matlab
- Can do simple problems by integrating ODEs
- Issues:
  - mixture of continuous and sampled time
  - hybrid logic (conditions)
  - state machines
  - stiff systems, algebraic loops
  - systems integrated out of many subsystems
  - large projects, many people contribute different subsystems

Simulation environment



- Simulink by Mathworks
- Matlab functions and analysis
- Stateflow state machines

- Ptolemeus - UC Berkeley

- Block libraries

- Subsystem blocks developed independently

- Engineered for developing large simulation models

- Supports code generation

Model block development

- Look up around for available conceptual models
- Physics - conceptual modeling
- Science (analysis, simple conceptual abstraction) vs. engineering (design, detailed models - out of simple blocks)

Modeling uncertainty

- Modeling uncertainty:
  - unknown signals
  - model errors
- Controllers work with real systems:
  - Signal processing: data $\rightarrow$ algorithm $\rightarrow$ data
  - Control: algorithms in a feedback loop with ***a real*** system
- BIG question: Why controller designed for a model would *ever* work with a *real* system?
  - Robustness, gain and phase margins,
  - Control design model, vs. control analysis model
  - Monte-Carlo analysis - a fancy name for a desperate approach