Lecture 2 – Linear Systems

This lecture: EE263 material recap + some controls motivation

• Continuous time (physics)
• Linear state space model
• Transfer functions
• Black-box models; frequency domain analysis
• Linearization
Modeling and Analysis

This lecture considers

- Linear models. More detail on modeling in Lecture 7
- Simulation: computing state evolution and output signal
- Stability: does the solution diverge after some time?
- Approximate linear models
Linear Models

• Model is a mathematical representations of a system
  – Models allow simulating the system
  – Models can be used for conceptual analysis
  – Models are never exact

• Linear models
  – Have simple structure
  – Can be analyzed using powerful mathematical tools
  – Can be matched against real data using known procedures
  – Many simple physics models are linear
  – They are just models, not the real systems
State space model

- Generic state space model
  - is described by ODEs
  - e.g., physics-based system model
  - state vector \( x \)
  - observation vector \( y \)
  - control vector \( u \)

\[
\frac{dx}{dt} = f(x, u, t) \rightarrow \text{state evolution}
\]

\[
y = g(x, t) \rightarrow \text{observation}
\]

- Example: F16 Longitudinal Model

\[
\begin{align*}
\dot{V} & = -1.93 \cdot 10^{-2} V + 8.82 \alpha - 32.2 \theta - 0.58 q + 0.17 \delta_e \\
\dot{\alpha} & = -2.54 \cdot 10^{-4} V - 1.02 \alpha + 0.91 q - 2.15 \cdot 10^{-3} \delta_e \\
\dot{\theta} & = q \\
\dot{q} & = 2.95 \cdot 10^{-12} V + 0.82 \alpha - 1.08 q - 0.18 \delta_e
\end{align*}
\]

- \( x_1 \): velocity \( V \) [ft/sec]
- \( x_2 \): angle of attack \( \alpha \) [rad]
- \( x_3 \): pitch angle \( \theta \) [rad]
- \( x_4 \): pitch rate \( q \) [rad/sec]
- \( \delta_e \): elevator deflection [deg]

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EE392m - Spring 2005
Gorinevsky
Linear state space model

- Linear Time Invariant (LTI) state space model:
  \[
  \frac{dx}{dt} = Ax + Bu \quad \text{→ state evolution}
  \]
  \[
  y = Cx \quad \text{→ observations}
  \]

- Can be integrated analytically or numerically (simulation)
- Can be well analyzed: stability, response

Example: F16 Longitudinal Model

\[
\frac{dx}{dt} = \begin{bmatrix}
-1.93 \cdot 10^{-2} & 8.82 & -32.2 & -0.48 \\
-2.54 \cdot 10^{-4} & -1.02 & 0 & 0.91 \\
0 & 0 & 0 & 1 \\
2.95 \cdot 10^{-12} & 0.82 & 0 & -1.08
\end{bmatrix} \cdot x + \begin{bmatrix}
0.17 \\
-2.15 \cdot 10^{-3} \\
0 \\
-0.18
\end{bmatrix} \cdot u
\]

\[
y = \begin{bmatrix}
0 & 0 & 57.3 & 0
\end{bmatrix} \cdot x
\]
Integrating linear autonomous system

\[ \frac{dx}{dt} = Ax \]

- Matrix exponential
  - Can be computed in Matlab as \texttt{expm}(A)
  - The definition corresponds to integrating the ODE by Euler method

\[
x(T + \Delta t) \approx x(T) + \Delta tAx(t) = (I + \Delta tA)x(T)
\]

\[
x(T + n\Delta t) = (I + \Delta tA)^n x(T) \rightarrow \exp(A) \Delta t x(T)
\]

\[
\exp(At) = \lim_{\Delta t \to 0} (I + \Delta tA)^{t/\Delta t}
\]

Example:

\[
>> A = \begin{bmatrix} -1.93e-2 & 8.82 & -32.2 & -0.58 \\ -2.54e-4 & -1.02 & 0 & 0.91 \\ 0 & 0 & 0 & 1 \\ 2.95e-12 & 0.82 & 0 & -1.08 \end{bmatrix};
\]

\[
>> \text{expm}(A)
\]

\[
\text{ans} =
\begin{bmatrix}
0.9802 & 3.1208 & -31.8830 & -10.2849 \\
-0.0002 & 0.5004 & 0.0031 & 0.3604 \\
-0.0000 & 0.2216 & 1.0002 & 0.6709 \\
-0.0001 & 0.3239 & 0.0007 & 0.4773
\end{bmatrix}
\]
Integrating linear autonomous system

\[ \frac{dx}{dt} = Ax, \quad x(0) = x_0 \]

- Initial condition response

\[ x(t) = \exp(At)x_0 \]

Example:

```
%Take A, B, C from the F16 example
>> k = 0.02
>> G = A + k*B*C;
>> x0 = [0.1700; -0.0022; 0; -0.1800];
>> for j = 1:length(t);
    x(:,j)=expm(G*t(j))*x0; end;
```
Eigenvalues and Stability

- Consider eigenvalues of the matrix $A$
  \[
  \{\lambda_j\} = \text{eig}(A); \quad \text{det}(I\lambda - A) = 0
  \]

- Suppose $A$ has all different and nonzero eigenvalues, then
  \[
  A = V \cdot \text{diag}\{\lambda_j\} \cdot V^{-1}
  \]
  \[
  \exp(At) = V \cdot \text{diag}\{e^{\lambda_j t}\} \cdot V^{-1}
  \]
- The system solution is exponentially stable if $\text{Re} \lambda_j < 0$
- If $A$ has eigenvalues with multiplicity more than 1, things are a bit more complicated: Jordan blocks, polynomials in $t$
- Still the condition of exponentially stability is $\text{Re} \lambda_j < 0$

Example:

```matlab
% take A from % the F16 example
>> eig(A)
ans =
   -1.9125   -0.1519 + 0.1143i
   -0.1519 - 0.1143i
    0.0970
```

\( j \)

\( \lambda \)

\( \text{Re} \)

\( t \)
Input-output models

• Black-box models – describe system $P$ as an operator

Historically (50 years ago)
  – Black-box models $\rightarrow$ EE
  – State-space models $\rightarrow$ ME, AA
Linear System (input-output)

• Linearity

\[
\begin{align*}
    u_1(\cdot) & \xrightarrow{P} y_1(\cdot) &
    u_2(\cdot) & \xrightarrow{P} y_2(\cdot) \\
    a u_1(\cdot) + b u_2(\cdot) & \xrightarrow{P} a y_1(\cdot) + b y_2(\cdot)
\end{align*}
\]

• Linear Time-Invariant systems - LTI

\[
\begin{align*}
    u(\cdot - T) & \xrightarrow{P} y(\cdot - T)
\end{align*}
\]
Convolution representation

- Convolution integral
  \[ y(t) = \int_{-\infty}^{t} h(t - \tau)u(\tau)d\tau \]

- Impulse response
  \[ u(t) = \delta(t) \quad \Rightarrow \quad y(t) = h(t) \]

- Step response: \( u = 1 \) for \( t > 0 \)
  \[ g(t) = \int_{0}^{t} h(t - \tau)d\tau = \int_{0}^{t} h(\tau)d\tau \]
  \[ h(t) = \frac{d}{dt} g(t) \]
Impulse Response for State Space Model

\[
\frac{dx}{dt} = Ax + Bu \rightarrow \text{state evolution}
\]

\[
y = Cx \rightarrow \text{observation}
\]

- Impulse response for the state \( x \)
  \[
x(\Delta t) \approx \Delta t Ax(0) + B \Delta tu
  \]
  \[
x(\Delta t) \approx B \Delta tu = B
  \]
  \[
x(t) = \exp(At)B
  \]

- System impulse response
  \[
h(t) = C \exp(At)B
  \]

Example:
```matlab
>> A = [-0.3130  56.7000  0;
       -0.0139  -0.4260  0;
          0     56.7000  0];
>> B = [0.232; 0.0203; 0];
>> C = [0, 0, 1];
```

![Graphs showing impulse responses](image)
Formal transfer function

- Rational transfer function
  \( = \text{IIR (Infinite Impulse Response) model} \)
  - Broad class of input-output linear models

\[
a_1 \frac{d^m y}{dt^m} + a_2 \frac{d^{m-1} y}{dt^{m-1}} + \ldots + a_{m+1} y = b_1 \frac{d^n u}{dt^n} + b_2 \frac{d^{n-1} u}{dt^{n-1}} + \ldots + b_{n+1} u
\]

- Differentiation operator \( \frac{d}{dt} \rightarrow s \)

- Formal transfer function – rational function of \( s \)

\[
y = H(s) \cdot u = \frac{N(s)}{D(s)} \cdot u \quad N(s) = a_1 s^m + \ldots + a_m s + a_{m+1} \quad D(s) = b_1 s^n + \ldots + b_n s + b_{n+1}
\]

- For a causal system \( m \leq n \)
Poles, Impulse Response

\[ y = \frac{N(s)}{D(s)} \cdot u \]

- Zeros: \( N(s) = 0 \)
- Poles: \( D(s) = 0 \)

- Expand: \( D(s) = (s - p_1)^{M_1} \cdots (s - p_K)^{M_K} \)

Then
\[ y = \frac{N_1(s)}{(s - p_1)^{M_1}} \cdot u + \cdots + \frac{N_K(s)}{(s - p_K)^{M_K}} \cdot u \]

- Quasi-polynomial impulse response – see a textbook
\[ h(t) = \left(c_{1,1} t^{M_1-1} + \cdots + c_{1,M_1+1}\right)e^{p_1 t} + \cdots + \left(c_{K,1} t^{M_K-1} + \cdots + c_{K,M_K+1}\right)e^{p_K t} \]

- Example: \( \frac{d^2 y}{dt^2} = u \)

Transfer function: \( y = \frac{1}{s^2} \cdot u \)

Impulse response: \( h(t) = v_0 t + x_0 \)
Transfer Function for State Space

• Formal transfer function for a state space model

\[
\frac{dy}{dt} \rightarrow s \\
\begin{align*}
x(t) &= Ax + Bu \\
y &= Cx \\
y &= (sI - A)^{-1}B \cdot u
\end{align*}

H(s) = C(sI - A)^{-1}B

• Characteristic polynomial

\[N(s) = \det(sI - A) = 0\]

Poles \(\Leftrightarrow\) \(\det(sI - A) = 0 \Leftrightarrow\) eigenvalues

• Poles are the same as eigenvalues of the state-space matrix \(A\)

• For stability we need \(\text{Re} p_k = \text{Re} \lambda_k < 0\)
Laplace transform

• Laplace integral transform: \( x(t) \rightarrow \hat{x}(s) = \int_0^\infty x(t)e^{st} \, dt \)

• Laplace transform of the convolution integral yields

\[ \hat{y}(s) = H(s)\hat{u}(s) \quad H(s) = \int_0^\infty h(t)e^{st} \, dt \]

• Transfer function: \( H(s) \)
  – function of complex variable \( s \)
  – analytical in a right half-plane \( \Re s \geq a \)
  – for a stable system \( a \leq 0 \)
  – for an IIR model \( H(s) = \frac{N(s)}{D(s)} \quad \frac{dx}{dt} \rightarrow s\hat{x}(s) \)
Frequency decomposition

- Sinusoids are eigenfunctions of an LTI system: \( y = H(s)u \)

\[ e^{i\omega t} \rightarrow \text{LTI Plant} \rightarrow H(i\omega)e^{i\omega t} \]

\[ s(e^{i\omega}) \rightarrow \frac{d}{dt} e^{i\omega t} = i\omega e^{i\omega t} \]

- Frequency domain analysis

\[ \sum u_k e^{i\omega_k t} \rightarrow y = \sum u_k H(i\omega_k)e^{i\omega_k t} \]
Frequency domain description

- Frequency domain analysis

\[
    u = \frac{1}{2\pi} \int \tilde{u}(\omega) e^{-i\omega t} d\omega \quad \Rightarrow \quad y = \frac{1}{2\pi} \int H(i\omega) \tilde{u}(\omega) e^{-i\omega t} d\omega
\]

- Fourier transform – numerical analysis
- Laplace transform – complex analysis

\[
    \tilde{u}(\omega) = \int_{-\infty}^{\infty} u(t) e^{i\omega t} d\omega = \left(\int_{-\infty}^{\infty} u(t) e^{st} d\omega\right)_{s=i\omega} = \hat{u}(i\omega)
\]

\[u(t) = 0, \text{ for } t < 0\]
Continuous systems in frequency domain

\[
\tilde{x}(\omega) = \int_{-\infty}^{\infty} x(t) e^{i\omega t} dt
\]

\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{x}(\omega) e^{-i\omega t} d\omega
\]

\[
y(t) = \int_{-\infty}^{t} h(t-\tau)u(\tau) d\tau
\]

\[
H(s) = \int_{0}^{\infty} h(t) e^{-st} dt
\]

\[
\tilde{y}(\omega) = H(i\omega)\tilde{u}(\omega)
\]

- Fourier transform
  \([-\infty, \infty] \rightarrow [-\infty, \infty]\)
- Inverse Fourier transform
- I/O impulse response model
- Transfer function
- System frequency response
Frequency domain description

- Bode plots:
  \[ u = e^{i\omega t} \]
  \[ y = H(i\omega)e^{i\omega t} \]
- Example:
  \[ H(s) = \frac{1}{s - 0.7} \]
- \(|H|\) is often measured in dB
  - \([\text{dB}] = 20 \log_{10} M\)
Model Approximation

- Model structure – physics, computational
- Determine parameters from data
- Step/impulse responses are close $\leftrightarrow$ the input/output models are close
- Example – fit step response
- Linearization of nonlinear model
Black-box model from data

- Linear black-box model can be determined from the data, e.g., step response data, or frequency response

- Example problem: fit an IIR model of a given order
- This is called model identification
- Considered in more detail in Lecture 8
Linear PDE models

- Include functions of spatial variables
  - electromagnetic fields
  - mass and heat transfer
  - fluid dynamics
  - structural deformations
- Example: sideways heat equation

\[ \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} \]

\[ T(0) = u; \quad T(1) = 0 \]

\[ y = \frac{\partial T}{\partial x} \bigg|_{x=1} \]

\[ T_{inside} = u \quad T_{outside} = 0 \]

heat flux
Linear PDE System Example

- Heat transfer equation,
  - boundary temperature input $u$
  - heat flux output $y$
- Impulse response and step response
- Transfer function is not rational

$$\frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}$$

$u = T(0)$  \hspace{1cm} $T(1) = 0$

$$y = \left. \frac{\partial T}{\partial x} \right|_{x=1}$$

![Pulse Response Plot](image1)

![Step Response Plot](image2)
Impulse response approximation

- Approximating impulse and step responses by a low order rational transfer function model
- Higher order model can provide very accurate approximation
- Methods:
  - trial and error
  - sampled time response fit, e.g., Matlab’s prony
  - identification, Lecture 8
  - formal model reduction approaches - advanced

\[ H(s) = 0.01 \frac{0.08s^2 - 0.4s + 2.8}{s^2 + 0.34s + 0.03} \]
Validity of Model Approximation

• Why can we use an approximate model instead of the ‘real’ model?
• Will the analysis hold?
• The input-output maps of two systems are ‘close’ if the convolution kernels (impulse responses) are ‘close’

\[ y(t) = \int_{-\infty}^{t} h(t - \tau) u(\tau) \tau \]

• The closed-loop stability impact of the modeling error
  – Control robustness
  – Will be discussed in Lecture 9
Nonlinear map linearization

- Nonlinear - detailed model
- Linear - conceptual design model
- Differentiation, secant method

Example:
static map linearization
\[ y = f(u) \approx \frac{\Delta f}{\Delta u} (u - u_0) \]
Linearization Example: RTP

- RTP – Rapid Thermal Processing
- Major semiconductor manufacturing process

\[
\frac{dT}{dt} = bu - c_1(T^4 - T_F^4) - c_2(T - T_F)
\]

- \(T\) – part temperature
- \(u\) – IR heater power
- \(T_F\) – furnace temperature

- Stefan-Boltzmann law nonlinearity
- \(T_F\) is assumed to be constant
RTP, cont’d

\[
\frac{dT}{dt} = f(T) + bu
\]

\[
f(T) = -c_1(T^4 - T_F^4) - c_2(T - T_F)
\]

Linearize around a steady state point

\[
\frac{dT}{dt} = f_L(T) + bu
\]

\[
f_L(T) = a(T - T_*) + d
\]

\[
d = f(T_*) \quad a = f''(T_*) \approx \frac{f(T_* + \Delta) - f(T_*)}{\Delta}
\]

\[
b = 1000, \quad c_1 = 1.1 \cdot 10^{-10}, \quad c_2 = 0.8, \quad T_F = 300
\]
RTP, cont’d

\[ \dot{x} = ax + bu + d \]
\[ u = -kx \]

\[ x = T - T_\star \quad \text{Linear system with a pole} \]
\[ p = -(a + bk) \]

\[ T_\star = 1000, \; a = -1.7425, \; b = 1000, \; k = 0.01 \quad \Rightarrow \quad p = -11.7425 \]

Simulate performance:

![Simulation graph]

Linear model, \( d = 0 \)
Non-linear model
Nonlinear state space model linearization

• Linearize the r.h.s. map in a state-space model

\[
\dot{x} = f(x,u) \approx \frac{\Delta f}{\Delta x}(x - x_0) + \frac{\Delta f}{\Delta u}(u - u_0)
\]

\[
\dot{q} = Aq + Bv
\]

• Linearize around an equilibrium \( 0 = f(x_0,u_0) \)

• Secant method

\[
\left[ \frac{\Delta f}{\Delta x} \right]^j = \frac{f(x_0 + s_j, u_0) - f(x_0, u_0)}{d_j}
\]

\[
s_j = [0 \ldots d_j \ldots 0]
\]

• This is how Simulink computes linearization
Example: F16 Longitudinal Model

\[ \frac{dx}{dt} = f(x, u) \]

- State vector \( x \)
  - \( x_1 \) - velocity \( V \) [ft/sec]
  - \( x_2 \) - angle of attack \( \alpha \) [rad]
  - \( x_3 \) - pitch angle \( \theta \) [rad]
  - \( x_4 \) - pitch rate \( q \) [rad/sec]
- Control input
  - \( u \) - elevator deflection \( \delta_e \) [deg].

\[
\begin{bmatrix}
V \\
\alpha \\
q \\
\theta
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{m} (F_x \cos \alpha + F_z \sin \alpha) \\
\frac{1}{mV} (-F_x \sin \alpha + F_z \cos \alpha) + q \\
\frac{M_y}{I_y} \\
q
\end{bmatrix}
\]

For more detail see: *Aircraft Control and Simulation* by Stevens and Lewis
Nonlinear Model of F16

\[
\frac{dx}{dt} = f(x, u) \quad \rightarrow \text{state evolution}
\]

\[y = g(x) \quad \rightarrow \text{observation}\]

- Aircraft models are understood by groups of people.
- Could take many man-years worth of effort.
- Aerodynamics model is based on empirical data.
- \(f(x, u)\) available as a computational function can be used without a deep understanding of the model.
- The nonlinear model can be used for simulation, or linearized for analysis.
Linearized Longitudinal Model of F16

• Assume trim condition

\[ x_0 = \begin{bmatrix} V_0 \\ \alpha_0 \\ q_0 \\ \theta_0 \end{bmatrix} = \begin{bmatrix} 500 \\ 0.0393 \\ 0 \\ 0.0393 \end{bmatrix} \]

- velocity \( V \) [ft/sec]
- angle of attack \( \alpha \) [rad]
- pitch rate \( q \) [rad/sec]
- pitch angle \( \theta \) [rad]

• Linearize the nonlinear function \( f(x,u) \) by a finite difference method (secant method). Step = [1 0.001 0.01 0.001]

\[
A = \frac{\Delta f}{\Delta x} = \begin{bmatrix}
-1.93 \cdot 10^{-2} & 8.82 & -32.2 & -0.48 \\
-2.54 \cdot 10^{-4} & -1.02 & 0 & 0.91 \\
0 & 0 & 0 & 1 \\
2.95 \cdot 10^{-12} & 0.82 & 0 & 1.08
\end{bmatrix}
\]

\[
B = \frac{\Delta f}{\Delta u} = \begin{bmatrix}
0.17 \\
-2.15 \cdot 10^{-3} \\
0 \\
-0.18
\end{bmatrix}
\]

• These are the matrices we considered in the linear F16 model example
Simulation-based validation

- Simulate with nonlinear model, compare with linear model results
- Doublet response

(a) Velocity $V$

(b) Angle of attack $\alpha$

(c) Pitch rate $q$

(d) Pitch angle $\theta$
LTI models - summary

• ODE model
• State space linear model
• Linear system can be described by impulse response or step response
• Linear system can be described by frequency response = Fourier transform of the impulse response
• Linear model approximations can be obtained from more complex models
  – Approximation of a linear model response
  – Linearization of a nonlinear model