Lecture 7 - SISO Loop Analysis

SISO = Single Input Single Output

Analysis:
• Stability
• Performance
• Robustness
ODE stability

- Lyapunov’s mathematical stability theory - nonlinear systems
  - stability definition
  - first (direct) method
    - exponential convergence
  - second method: Lyapunov function
    - generalization of energy dissipation

- Lyapunov’s exponent
  - dominant exponent of the convergence
  - for a nonlinear system
  - for a linear system defined by the poles
Stability: poles

\[
\begin{align*}
\dot{x} &= Ax + Bu & y &= H(s) \cdot u \\
y &= Cx + Du & H(s) &= C(I - s^{-1} A)^{-1} B + D
\end{align*}
\]

• Characteristic values = transfer function poles
  – l.h.p. for continuous time
  – unit circle for sampled time

• I/O model vs. internal dynamics

\[
H(s) = \frac{N(s)}{D(s)} = \frac{g_1}{s - p_1} + \ldots + \frac{g_n}{s - p_n} + g_0
\]
Stability: closed loop

The transfer function poles are the zeros of

\[ 1 + P(s)C(s) \]

Watch for pole-zero cancellations!

Poles define the closed-loop dynamics (including stability)

Algebraic problem, easier than state space sim

\[ u = -C(s)e; \quad e = y - y_d \]

\[ e = \left[ 1 + P(s)C(s) \right]^{-1} y_d \]
Servomotor Example

- The control goal is to track the position setpoint $y_d$
- Use PID control $\tau_D = 0.01$

$$C(s) = k_P + k_I \frac{1}{s} + k_D \frac{s}{\tau_D s + 1}$$

Transfer function

$$y = \frac{G}{(1 + T_F s)(1 + T_I s)(s + T_J s^2)} u$$

$T_J = 0.1\,\text{sec}$, $T_I = 0.02\,\text{sec}$, $T_F = 0.001\,\text{sec}$, $G = 1$
Servomotor Example

- PID controller: \( k_P = 10; k_I = 100; k_D = 0.1; \tau_D = 0.01 \)

\[
S(s) = [1 + P(s)C(s)]^{-1}
\]

- Stability

```matlab
>> S = feedback(1,C*P);
>> [z,p,k] = zpkdata(S);
>> plot(p)
```
Stability

For linear system poles describe stability

• … almost, except the critical stability
• For nonlinear systems
  – linearize around the equilibrium
  – might have to look at the stability theory - Lyapunov
• Orbital stability:
  – trajectory converges to the desired
  – the state does not - the timing is off
    • spacecraft
    • FMS, 3-D trajectories without aircraft arrival time
Performance

- Need to describe and analyze performance so that we can design systems and tune controllers
- What is the performance index?
- There are usually many conflicting requirements
- Engineers look for a reasonable trade-off

\[
u = -\left( k_D \frac{s}{\tau_D s + 1} + k_P + \frac{k_I}{s} \right) e
\]
Performance: Example

- Selecting optimal \( b \) in the Watt’s governor - HW Assignment 1
Performance - poles

- Steady state error: study transfer functions at $s=0$.
- Step/pulse response convergence, dominant pole

$$a = \min \left\{ \Re p_j \right\}_{j=1}^n$$

- Caution! Fast response (poles far to the left) may lead to peaking

$C + Ae^{-at}$ dominant exponent

fast response

slow response
Performance - step response

• Step response shape characterization:
  - overshoot
  - undershoot
  - settling time
  - rise time

stead state error
settling error
Performance - quadratic index

- Quadratic performance
  - response, in frequency domain

\[ J = \int_{t=0}^{\infty} \left| y(t) - y_d(t) \right|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\tilde{e}(i\omega)|^2 d\omega = \frac{1}{2\pi} \int |S(i\omega)|^2 \frac{1}{\omega^2} d\omega \]

- For \( y_d(t) \) a zero mean random process with spectral power \( Q(i\omega) \)

\[ J = E\left( \int_{t=0}^{\infty} \left| y(t) - y_d(t) \right|^2 dt \right) = \frac{1}{2\pi} \int |S(i\omega)|^2 Q(i\omega) d\omega \]

- For \( Q(i\omega) = 1 \), this is just Parceval’s theorem
Servomotor example

- Step response

\[ J = \int_{0}^{\infty} \left| y(t) - y_d(t) \right|^2 dt \]

- Quadratic index

\[
>> \ T = \text{feedback}(C*P,1); \\
>> \ \text{step}(T) \\
>>
\]

\[
>> \ T = \text{feedback}(C*P,1); \\
>> \ \text{dt} = 0.01; \\
>> \ \text{y} = \text{step}(T,0:\text{dt}:0.1); \\
>> \ J = \text{sum}((\text{y}-1).\wedge2*\text{dt}) \\
J = 0.0081
\]
Sensitivities

\[
y = S(s)d + T(s)y_d
\]

\[
S(i\omega) = \frac{1}{1 + L(i\omega)}
\]

- **Feedback sensitivity**
  - \(|S(i\omega)|<<1\) for \(|L(i\omega)|>>1\)
  - \(|S(i\omega)|\approx1\) for \(|L(i\omega)|<<1\)
  - can be bad for \(|L(i\omega)| \approx 1\) - ringing, instability

\[
S(s) = \left[1 + P(s)C(s)\right]^{-1}
\]

\[
L(s) = P(s)C(s)
\]

\[
S_{FF}(i\omega) = 1
\]

- **Feedforward sensitivity**
  - good for any frequency
  - never unstable

\[
y = d + F(s)P(s)y_d
\]
Transfer functions in control loop

disturbance
feedforward
noise
reference

d
v
n
y_d

Plant
\( P(s) \)

Controller
\( C(s) \)

\[ e = S(s)d - S(s)y_d + T(s)n + S_y(s)v \]
\[ y = S(s)d + T(s)y_d + T(s)n + S_y(s)v \]
\[ u = -S_u(s)d + S_u(s)y_d + S_u(s)n + T(s)v \]
Transfer functions in control loop

\[
\begin{align*}
e &= y - y_d + n \\
y &= P(s)(u + v) + d \\
u &= -C(s)e
\end{align*}
\]

\[
e = S(s)d - S(s)y_d + T(s)n + S_y(s)v \\
y = S(s)d + T(s)y_d + T(s)n + S_y(s)v \\
u = -S_u(s)d + S_u(s)y_d + S_u(s)n + T(s)v
\]

Sensitivity \( S(s) = \left[1 + P(s)C(s)\right]^{-1} \)

Complementary sensitivity \( T(s) = \left[1 + P(s)C(s)\right]^{-1} P(s)C(s) \)

Noise sensitivity \( S_u(s) = \left[1 + P(s)C(s)\right]^{-1} C(s) \)

Load sensitivity \( S_y(s) = \left[1 + P(s)C(s)\right]^{-1} P(s) \)

\[
S(s) + T(s) = 1
\]
Sensitivity requirements

\[ e = S(s)d - S(s)y_d + T(s)n + S_y(s)v \]
\[ y = S(s)d + T(s)y_d + T(s)n + S_y(s)v \]
\[ u = -S_u(s)d + S_u(s)y_d + S_u(s)n + T(s)v \]

\[ S(i\omega) = \frac{1}{1 + P(i\omega)C(i\omega)} \]
\[ S_y(i\omega) = \frac{P(i\omega)}{1 + P(i\omega)C(i\omega)} \]
\[ S_u(i\omega) = \frac{C(i\omega)}{1 + P(i\omega)C(i\omega)} \]

- Disturbance rejection and reference tracking
  - \(|S(i\omega)| << 1\) for the disturbance \(d\)
  - \(|S_y(i\omega)| << 1\) for the input ‘noise’ \(v\)

- Limited control effort
  - \(|S_u(i\omega)| << 1\) conflicts with disturbance rejection where \(|P(i\omega)| < 1\)

- Noise rejection
  - \(|T(i\omega)| << 1\) for the noise \(n\), conflicts with disturbance rejection
Servomotor example - sensitivities

Output disturbance

Setpoint tracking

Output noise

Feedforward

EE392m - Spring 2005
Gorinevsky
Control Engineering
7-18
Robustness

• A controller works for a model.
• Will it work for a real system?
• Can check that controller works for a range of different models and hope that the real system is covered by this range.
Robustness

- Additive uncertainty

\[
\begin{align*}
\Delta & \rightarrow \Delta \\
T(s) & \rightarrow C(s) \\
P(s) & \rightarrow y(t) \\
S_u(s) & \rightarrow u(t)
\end{align*}
\]

Condition of robust stability

\[
\left\| \frac{C(i\omega)}{1 + P(i\omega)C(i\omega)} \right\| \left\| \Delta(i\omega) \right\| < 1
\]

Small Gain Theorem: loop gain < 1 \(\rightarrow\) stability

- Multiplicative uncertainty

\[
\begin{align*}
\Delta & \rightarrow \Delta \\
T(s) & \rightarrow C(s) \\
P(s) & \rightarrow y(t) \\
u(t) & \rightarrow u(t)
\end{align*}
\]

Condition of robust stability

\[
\left\| \frac{P(i\omega)C(i\omega)}{1 + P(i\omega)C(i\omega)} \right\| \left\| \Delta(i\omega) \right\| < 1
\]
Nyquist stability criterion

- Homotopy “Proof”
  - $G(s)$ is stable, hence the loop is stable for $\gamma=0$. Gradually increase $\gamma$ to 1. The instability cannot occur unless $\gamma G(i\omega) + 1 = 0$ for some $0 \leq \gamma \leq 1$.
  - $|G(i\omega_{180})| < 1$ is a sufficient condition

- Subtleties: r.h.p. poles and zeros
  - Formulation and real proof using the argument principle, encirclements of -1
  - stable $\rightarrow$ unstable $\rightarrow$ stable as $0 \rightarrow \gamma \rightarrow 1$

Compare against Small Gain Theorem:
Gain and phase margins

- Loop gain
  \[ L(s) = P(s)C(s) \]
  \[ S(s) = [1 + L(s)]^{-1} \]

- Nyquist plot for \( L \)
  - at high frequency \( |L(i\omega)| << 1 \)
Gain and phase margins

- Bode plots
Servomotor example

- Gain crossover at 107 rad/s

- Phase crossover at 399 rad/s