Chapter 3

Solving for Voltage and Current

Nodal Analysis

If you know Ohm's Law, you can solve for all the voltages and currents in simple resistor circuits, like the one shown below. In this chapter, we'll cover the basic strategy of solving circuits, which is called *nodal analysis*. It consists of writing KCL (the sum of the current flowing into a node is zero) for each node in the circuit, using the node voltages and the device current equations to find the currents through each device. This chapter we will use resistors, current sources and voltage sources in our examples which were given in Chapter 2. By solving this system of equations, we can find the voltage at each node and the current between each pair of nodes. This basic method works for all circuits. It works by converting the circuit into a linear algebra problem (if there are not diodes involved. With diodes the problem is no longer a linear problem) which can be solved by many programs (for example Matlab). An example of a circuit you will be able to solve using this method is shown in Figure 3.1.

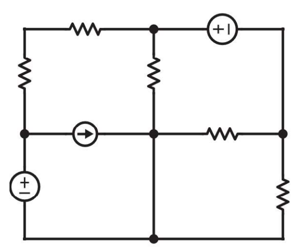


Figure 3.1: Example of a circuit we can solve using nodal analysis

We could stop there, but this method doesn't provide much insight about the effect each device has on the behavior of the circuit, which is important in the making (design) process. When you are designing a circuit you are really choosing the elements to generate the right behavior. To address this problem, after we describe nodal analysis we will provide a number of techniques that allow you to simplify the network you need to solve. Using these techniques generally simplifies the network to the point that you can solve them in your head, which gives you a much better understanding of what the component values need to be to get the effect you want. These simplification techniques include series and parallel reductions, voltage and current dividers, equivalent circuits, and superposition. Using these techniques you should be able to analyze a circuit without needing a computer. They are also very helpful when using non-linear devices like diodes, or transistors (we will introduce them in the next chapter).

3.1 What are nodes and node voltages

As we described in Section 1.3.1, in a circuit, a node is a place where two or more devices (resistors, diodes, batteries, etc) meet. Remember in our circuit model abstraction the lines that connect devices have no resistance (they are perfect conductors) so the voltage drop along these lines must be 0V. Thus we collapse the entire wire (which can have many segments) into a single node, since the voltage at any point on that line will be the same as the voltage at any other point on that line.

Let's try to identify the nodes in the circuit shown in Figure 3.1. A good way to find nodes is to pick a point on a wire, and follow the wire in all directions possible until you run into another component. One node might only connect two components, or it might connect many. Figure 3.2 shows each of the nodes.

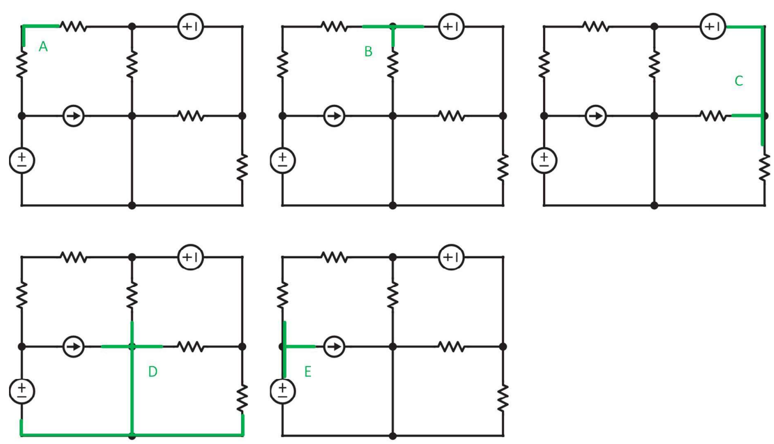


Figure 3.2: Identifying the nodes of the circuit

As we described in Chapter 1, since voltage is a measure of potential difference rather than an absolute number, a single node can't really have a voltage. The voltage of a node has to be measured relative to another node: only voltage differences are well defined. To get around this problem, in each circuit, we choose a node, referred to as **ground**, or **gnd**, to act as the reference node that we will use to measure the voltage of the other nodes. When we talk about a node voltage, we are really measuring the voltage difference between that node and the "gnd" node. Using this definition it is clear that the voltage of the "gnd" node must be 0V: the voltage difference between a node and itself is always zero. In our circuit diagram we use the symbol shown below to indicate the ground node.

We can choose any node we want to serve as ground. Moving ground to a different node will change the voltage you report for that node (since it is now measured relative to a different node), but it will not change the voltage difference between the nodes. In the figure below, notice how we chose a different ground node for each circuit, so the voltage at each node looks different in each figure. However, for each circuit, the voltage difference between nodes A and B is 2 V, between B and C is 3V, and between A and C is 5V. Since the current through a device depends on the voltage **difference** between its two terminals, the choice of reference voltage doesn't change any of the current flowing in the circuit. All of this just shows that voltage is relative: changing which node we call ground never changes the voltages across the devices or the current flowing through them.

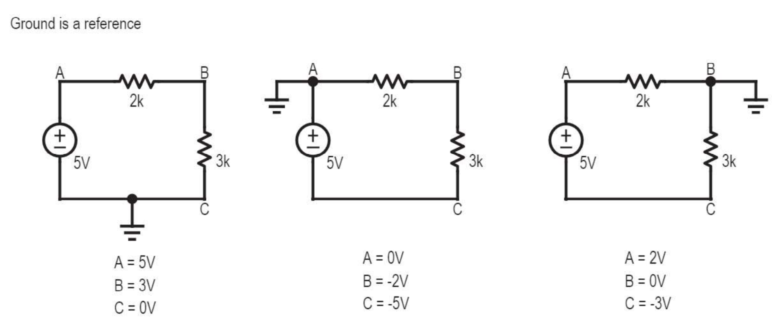


Figure 3.3: Grounding a circuit at different nodes

By convention, many circuit designers choose the lowest node in the schematic to serve as ground. Also by convention people tend to draw circuits so nodes with lower voltages on the bottom of their pictures, so the ground node is often the lowest voltage node. Another strategy is to make the ground node the node with the most connections to other devices, since, as we will see next, this will simplify nodal analysis. For example, in our circuit from the first figure, we might choose node D, as shown in Figure 3.4 since that node connects five components.

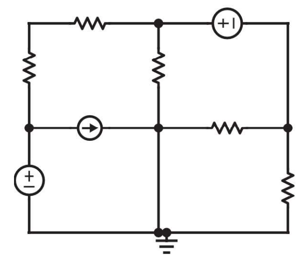


Figure 3.4: Choosing a ground for our first circuit example

However, if we wanted to choose another node, like B, we could do that, too. It often helps to draw ground on the lowest node so if we make B ground we can even re-draw the circuit by rotating it so our ground node is at the bottom, as illustrated in Figure 3.5

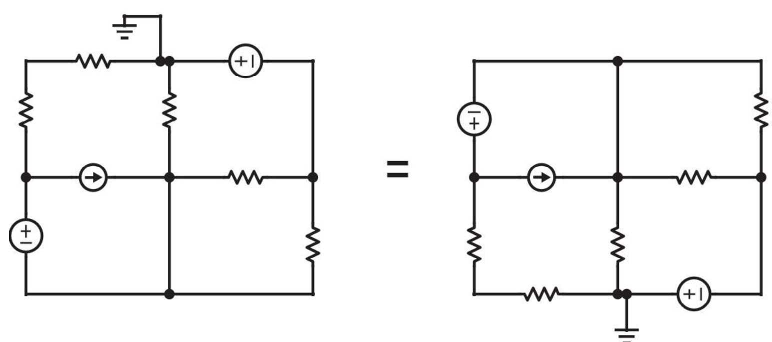


Figure 3.5: Choosing a ground for our first circuit example

In general, you can choose any node you want to be your ground node. Just make sure you mark your ground node clearly whenever you're solving a circuit.

3.2 Nodal Analysis

Let's take the circuit we have been looking at, set node D to be ground and try to find the voltage at node B. At first this seems complex, since at first we don't know what any of the voltages are (except node D, which we know will be at 0V), and they all seem to depend on each other. Fortunately there is a very simple procedure you can use to solve any circuit you run across. All you do is write KCL equations for each of the nodes in the circuit skipping the ground node. This will give you one equation for each node, and the only variables (things you don't know the value of) these equation depend on are the node voltages. So if we have 'n' nodes with unknown voltages, we will end up with 'n' equations in 'n' variables, which means if the device equations are linear, and resistors, voltage and current sources are linear, linear algebra we can solve these equations to find the node voltages. This simple procedure is called nodal analysis.

Rather than starting with this complex circuit, let's start with a simpler example. In Figure 3.6, we want to find the voltages at nodes A and B and the currents i_1 and i_2. We start writing KCL equations for node A and run into our first problem: a voltage source can supply any current. This makes the current equation for this node less than helpful. Fortunately since the device is a voltage source, we know the voltage across it is fixed (5V in this example). Thus we can find the voltage at node A without writing the current equation there. $V_A = 5$V, sine the other side is connected to gnd. Now we can write the KCL equation for node B:

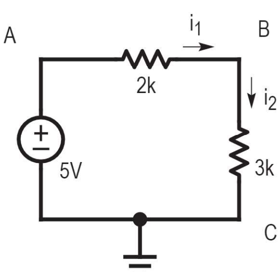


Figure 3.6: A simple circuit

$$i_1 - i_2 = 0$$

(i_1 flows into B, and i_2 flows out of B). The currents i_1 and i_2 can be found from Ohm's Law:

$$i_1 = \frac{5V - V_B}{2k\Omega}$$

$$i_2 = \frac{V_B}{3k\Omega}$$

In this step, is it very important that you calculate the current for the reference directions you choose in the problem. To find the current that flows in a particular direction, move in that direction through the device. The node you exit from moving in that direction should always be subtracted from the node you start at; e.g node B should be subtracted from node A for i_1.

This equation lets you solve for V_B:

$$\frac{5V - V_B}{2k\Omega} - \frac{V_B}{3k\Omega} = 0$$

Multiply both sides by 6kΩ:

$$15V - 3V_B - 2V_B = 0$$

$$15V = 5V_B$$

$$V_B = 3V$$

Let's try a more complicated example. Inthe circuit shown in Figure 3.7, we want to solve for the voltages at nodes A - D, and the currents $i_1 - i_5$. It sounds complicated, but we can use KCL and KVL just like we did in the example above.

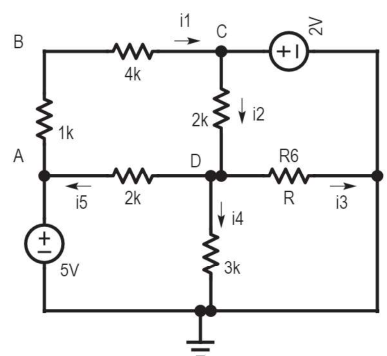


Figure 3.7: A more complex circuit

Let's start with node A. Again, only a voltage source is connected between ground and node A so V_A = 5V. We can do the same thing with node C - V_C = 2V.

In this circuit we were lucky that the voltage sources connected to gnd, so we know the voltage at the both sides of the voltage source. But you can have a voltage source that is not connected to ground. Suppose in the circuit above, node D was chosen as the reference node, and the node labeled gnd is now node E. While we have a voltage source, now we don't know what the voltage is at either node E, V_E, or node C, V_C. In this case you form what's called a **supernode**. This doesn't really mean anything other than that you can form a simple substitution equation. We don't know V_C because we don't know V_E, but we *do* know that $V_C = V_E + 2$, because the voltage source sets the voltage rise between them. So we don't really have two unknown node voltages, we only have one unknown. So we can still remove the variable V_C and replace it with $V_E + 2$. And while we don't know the current that flows through the voltage source we know the that that the current that flows into it at node C must flow out of it at node E. Using these constraints we couple the two node equations into a single supernode equation for nodes C and E.

Going back to our actual circuit, while the circuit has 4 nodes other than gnd, it only has two nodes with unknown voltages. So we need to write two KCL equations. Starting with node B, we know that the current through the 1kΩ resistor between A and B is the same as the current through the 4kΩ resistor between B and C, or i_1, which gives:

$$\frac{V_A - V_B}{1\,\text{k}\Omega} - \frac{V_B - V_C}{4\,\text{k}\Omega} = 0$$

$$\frac{5V - V_B}{1\,\text{k}\Omega} - \frac{V_B - 2V}{4\,\text{k}\Omega} = 0$$

Let's now move to node D.

$$i_2 = i_3 + i_4 + i_5$$

$$\frac{V_C - V_D}{2k\Omega} = \frac{V_D}{4k\Omega} + \frac{V_D}{4k\Omega} + \frac{V_D - V_A}{1k\Omega}$$

$$\frac{2V - V_D}{2k\Omega} = 2\frac{V_D}{4k\Omega} + \frac{V_D - 5V}{1k\Omega}$$

Using both equations we can solve for the two node voltages.[1] Multiplying both equations by $4\,\text{k}\Omega$ gives

$$20V - 4 \cdot V_B - V_B + 2V = 0$$

$$5 \cdot V_B = 22V$$

$$V_B = 4.4V$$

$$4V - 2V_D = 2V_D + 4V_D - 20V$$

$$8V_D = 24V$$

$$V_D = 3V$$

Now that we have the voltage at B and D, we can find all the currents:

$$i_1 = \frac{5V - V_B}{1k\Omega} = \frac{5V - 4.4V}{1k\Omega} = 0.6mA$$

[1] You might notice that in this problem you can use the first equation to solve for V_B, and the second equation to solve for V_D which is pretty unusual for these type of problems. More generally both equations will have both node voltages.

$$i_2 = \frac{V_C - V_D}{2k\Omega} = \frac{2V - 3V}{2k\Omega} = -0.5mA$$

$$i_3 = \frac{V_D}{4k\Omega} = 0.75mA$$

$$i_4 = \frac{V_D}{4k\Omega} = 0.75mA$$

$$i_5 = \frac{V_D - V_A}{1k\Omega} = -2mA$$

We can check our original KCL equation to make sure that it holds:

$$i_2 = i_3 + i_4 + i_5$$

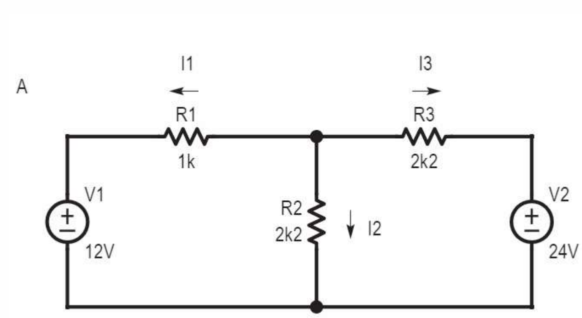
$$-0.5mA = 0.75mA + 0.75mA - 2mA$$

$$-0.5mA = -0.5mA$$

Notice that we wound up with negative currents for i_2 and i_5. This is completely fine! It just means that the direction we assumed the current would flow is backwards from where it's actually flowing. In circuits, everything is relative. There is no such thing as an absolute voltage or an absolute current direction. This is why it's so important to carefully mark the direction of your current, the positive and negative terminals of your voltage, and the location of your ground.

In nodal analysis the number of equations is proportional to the number nodes, and the complexity of the equations depends on the number of elements. As we will see in this next section we can use local rules to reduce both of these variables which can make the circuit much simpler to analyze. We will start with simple rules for combining two devices into a single device.

Problem 3.1
What are the currents I1, I2 and I3 (the current through resistors R1, R2, and R3 respectively)?



3.3 Series and Parallel Resistors

While nodal analysis always works, there are way you can simplify the math that you need to do. These steps are in some sense optional, since they just reduce the number of devices and/or nodes you need to analyze the circuit, but are often very helpful to understand what is happening in the circuit.

3.3.1 Series Connections

As we mentioned in Section 1.7, when analyzing circuits, there are some connections between devices that have special constraints that we can make use of. These occur when resistors are connected in series or in parallel. A **series** connection refers to one in which a terminal of the first device is connected *only* to a terminal of the second device. Because these two devices are the only devices connected at this node, the current through each must be the same. This follows from KCL: the sum of currents at the node between the two device must be equal to zero, so the same current must flow out of one device and into the other. Figure 3.8 shows a situation where two devices are in series, and a situation where none of the three devices are in series.

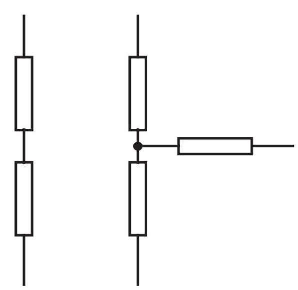


Figure 3.8: Left: Two devices in series; Right: No devices in series

When two resistors are connected in series, the voltage across the series combination is just going to be the sum of the voltage across each resistor. But this voltage is $i \cdot R_1$ and $i \cdot R_2$, and the current is the same in the two resistors. So the sum of the voltage across these two devices will be the same as the voltage across a single resistor of (R_1 + R_2). This is shown in Figure 3.9

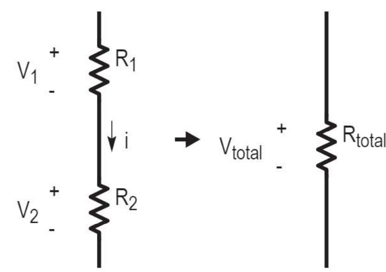


Figure 3.9: Combining resistors in series

$$V_1 = i * R_1$$

$$V_2 = i * R_2$$

$$V_{total} = V_1 + V_2 = iR_1 + iR_2 = i(R_1 + R_2) = iR_{total}$$

$$R_{total} = R_1 + R_2$$

The big advantage of doing this simplification is that you remove a node from the circuit, which removes one of the variable you need to solve for and one equation from your nodal analysis. While makes the linear algebra easier, its real advantage is when it makes the algebra so simple that you can solve it without really doing any linear algebra (which is true for most of the problems you need to solve).

If you have many resistors in series, you can repeatedly apply this method to combine a resistor to the previous equivalent resistors, so this equation generalizes to:

$$R_{total} = \sum_{n=1}^{N} R_n$$

Notice from this equation, that the total resistance along a path always increases as you add another resistor in series with it. Checking that this is true is always a good way to check your work.

3.3.2 Parallel Connections

A parallel connection refers to one in which the two terminals of the first component are connected to different terminals of the second component. Other components can also be connected to each terminal. Components connected in parallel have the same voltage drop across them. This follows from KVL: if we follow the loop around the two components connected in parallel, the total voltage must sum to zero, which means that each component must have the same voltage drop across it. The currents are not necessarily equal. Some examples are shown in Figure 3.10

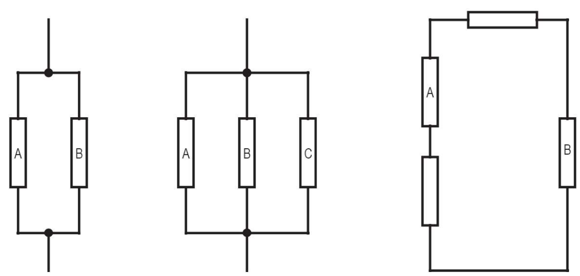


Figure 3.10: Left: A and B are in parallel; Center: A, B and C are in parallel; Right: A and B are not in parallel

When two resistors are connected in parallel, the equivalent resistance is equal to $\frac{1}{\frac{1}{R_1}+\frac{1}{R_2}}$. This is illustrated in Figure 3.11. We can find this by solving Ohm's Law across the two parallel components:

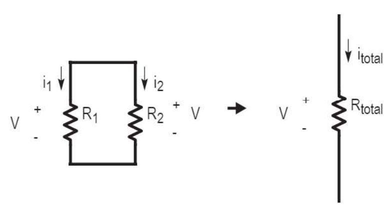


Figure 3.11: Combining resistors in series

$$V_{total} = i_1 R_1 = i_2 R_2$$
$$i_1 = \frac{V_{total}}{R_1}; i_2 = \frac{V_{total}}{R_2}$$
$$i_{total} = i_1 + i_2 = V_{total}(\frac{1}{R_1} + \frac{1}{R_2})$$
$$R_{total} = \frac{V_{total}}{i_{total}} = (\frac{1}{R_1} + \frac{1}{R_2})^{-1} = \frac{R_1 \cdot R_2}{R_1 + R_2}$$

This equation can be generalized for multiple resistors in parallel:

$$R_{total} = (\sum_{n=1}^{N} \frac{1}{R_n})^{-1}$$

From this equation, the equivalent resistance of resistors in parallel will be smaller than any of the individual resistors. With parallel connections, the current has more possible paths to take, which means that the ability of the circuit to conduct (conductance) increases, and resistance, which is the reciprocal of conductance, decreases.

3.3.3 Combining Series and Parallel Resistors

Let's try out what we know about series and parallel resistors to solve for the equivalent resistance between A and B in the circuit shown in Figure 3.12.

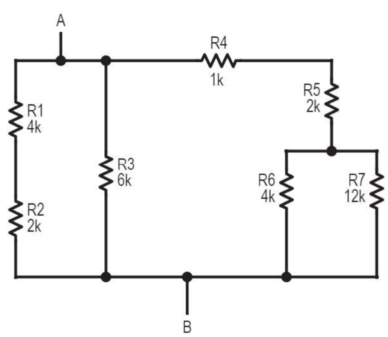


Figure 3.12: Find the equivalent resistance between points A and B

Let's start by identifying what we know are resistors in series and parallel. R_1 and R_2 are clearly in series, since one terminal of R_1 is connected directly to a terminal of R_2, with nothing else in the middle. Similarly R_4 and R_5 are in series. We know that we can combine two resistors in series into one resistor by simply adding the resistances, resulting in the circuit in Figure 3.13.

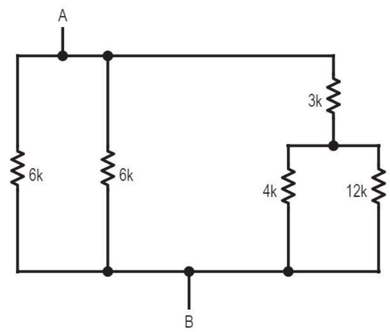


Figure 3.13: Find the equivalent resistance between points A and B - series resistances combined

Now let's try simplifying some parallel resistors. We can see that R_6 and R_7 are in parallel, since both of their terminals are connected to each other. We can use our parallel resistance equation, $R_{total} = \frac{R_6 \cdot R_7}{R_6 + R_7}$, to combine these two resistors into a single 3kΩ resistor. Similarly, R_3 is now in parallel with the combined R_1 and R_2. When two resistors of the same value are parallel to each other, the resulting equivalent resistance is just equal to half of the original resistor value (you can see this by solving the original equation: $\frac{R^2}{2R} = \frac{R}{2}$). Therefore, we can combine the left two branches of the circuit into a single 3kΩ resistor. The result is in Figure 3.14.

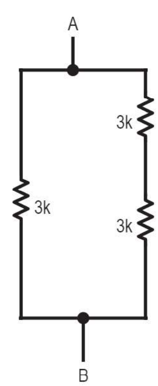
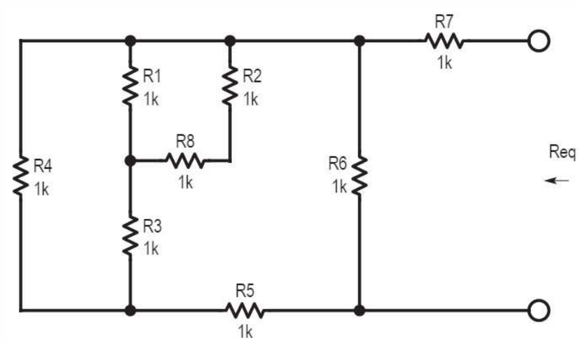


Figure 3.14: Find the equivalent resistance between points A and B - parallel resistances combined

This resulting circuit is pretty simple. The two resistors on the right are in series, so we can just add them to get a 6kΩ resistor. Then we have a 3kΩ in parallel with a 6kΩ, so we can use our parallel resistance equation to get an overall resistance of 2kΩ. We've now simplified the circuit into what's shown in Figure 3.15.

Figure 3.15: Final combined resistance

Problem 3.2
Find the equivalent resistance of this entire circuit between the two indicated nodes. (Find R_{eq}.)



3.4 Voltage and Current Dividers

Sometimes, we have two resistors in series, but we need to find the voltage at the node in the middle of the circuit, or we have two resistors in parallel but we need to find the current through one of the resistors. In both these cases we can do the series or parallel reduction, find the current through the series devices or the voltage across the parallel devices, and then solve for the desired voltage or current. But because these situations arise often, we have a shortcut for analyzing these types of circuits. One of the reasons that they occur often is that they have a useful function. These circuit produce an output voltage or current that is a fraction of the input voltage or current. Thus these circuits are called voltage and current dividers.

3.4.1 Voltage Dividers

In a simple voltage divider made of two resistors connected in series, we want to solve for the voltage drop across each resistor. We will start by solving for the voltage drop across the lower resistor. If we assume that the lower resistor is connected to ground, this is equivalent to the voltage at the intermediate node, V_{mid}. Since there is only one unknown in this circuit, V_{mid} we write KCL for this node:

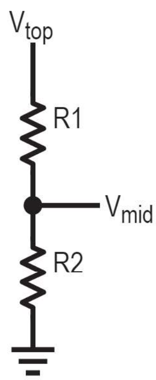


Figure 3.16: A voltage divider

$$\frac{V_{top}}{R_1} - \frac{V_{mid}}{R_1} = \frac{V_{mid}}{R_2}$$

$$V_{mid}(\frac{1}{R_1}) + \frac{1}{R_2} = \frac{V_{top}}{R_1}$$

$$V_{mid} = V_{top}\frac{\frac{1}{R_1}}{\frac{1}{R_1}+\frac{1}{R_2}} = V_{top}\frac{R_2}{R_1+R_2}$$

Notice that the output voltage is a fraction of the input voltage, and that fraction is the value of the bottom resistor, over the total resistance of the chain. This makes sense, since the current is going to be set by the total resistance. If most of the series resistance is from the bottom resistor, it will have most of the voltage across it. But as this resistance gets smaller (compared to the total), the voltage across the resistor gets smaller too.

If we want to solve for the voltage across R_1 instead, we would need to find $V_{top} - V_{mid}$. We can then rewrite the equations as follows:

$$V_{mid} = V_{top}\frac{R_2}{R_1+R_2}$$

$$V_{top} - V_{mid} = V_{top} - V_{top}\frac{R_2}{R_1+R_2} = V_{top}(1 - \frac{R_2}{R_1+R_2})$$

$$V_{top} - V_{mid} = V_{top}\frac{R_1+R_2-R_2}{R_1+R_2}$$

$$V_{top} - V_{mid} = V_{top}\frac{R_1}{R_1+R_2}$$

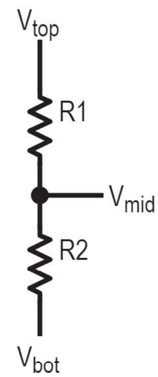


Figure 3.17: Another voltage divider

Question: Sometimes you will have a voltage divider situation by the bottom voltage won't be ground. Instead, let's assume it is V_{bot}. What do you do here?

While this situation seems more complicated than the grounded case, it is no harder to solve. You just need to remember that all voltages are relative. We know the voltage drop across R_2 is going to be:

$$(V_{top} - V_{bot}) \cdot \frac{R_2}{R_1 + R_2}$$

And this is equal the voltage $V_{mid} - V_{bot}$. So V_{mid} is just going to be:

$$V_{mid} = V_{bot} + (V_{top} - V_{bot}) \cdot \frac{R_2}{R_1 + R_2}$$

Note that if $V_{bot} = 0$, this equation simplifies to the same thing we had earlier. Basically, this is telling us that we need to multiply the resistance ratio by the voltage difference across the entire divider, then add the offset from the bottom voltage. In this case, the voltage across the top resistor is equal to:

$$V_{top} - V_{mid} = (V_{top} - V_{bot}) \frac{R_1}{R_1 + R_2}$$

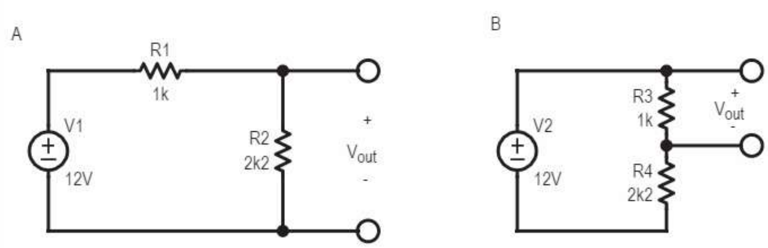
Again, this is the same equation we had earlier, but looking at the voltage difference between the resistors.

In general, the voltage across one of the components is equal to the resistance of that component divided by the sum of the two resistances. If we have multiple components connected in series between V_{top} and V_{bot}, this equation generalizes as shown below:

$$V_{R_x} = (V_{top} - V_{bot}) \frac{R_x}{\sum_{n=1}^{N} R_n} + V_{bot}$$

If R_1 and R_2 are multiple components connected in series or parallel, we can find the equivalent resistance of these two sets of components and solve from there.

Problem 3.3
Find V_{out} in the following two circuits.



3.4.2 Current Dividers

In a simple current divider made up of two resistors connected in parallel, we want to solve for the current through each resistor. In a parallel circuit, the voltage drop across each component is the same. We can use this to write equations for the circuit as shown below:

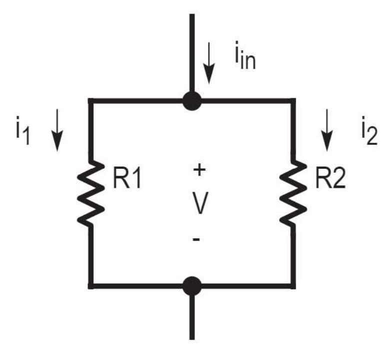


Figure 3.18: Another voltage divider

$$V_{total} = i_1 R_1 = i_2 R_2$$

$$i_2 = i_1 \frac{R_1}{R_2}$$

$$i_{total} = i_1 + i_2 = i_1 + i_1 \frac{R_1}{R_2}$$

$$i_{total} = i_1(1 + \frac{R_1}{R_2}) = i_1 \frac{R_1 + R_2}{R_2}$$

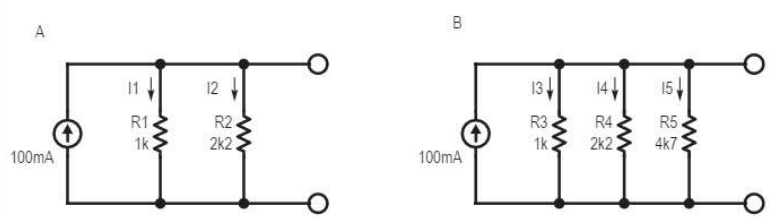
$$i_1 = i_{total} \frac{R_2}{R_1 + R_2}$$

We can flip these equations to find the current through the opposing resistor, $i_2 = i_{total} \frac{R_1}{R_1+R_2}$. This essentially tells us that the fraction of the original current through one resistor is equal to the other resistance, divided by the sum of the two resistances. Notice that equation looks similar the the voltage divider equation, for a current divider, the numerator is the opposite resistor, not the resistor we are measuring the current through. This makes sense, since as the resistance of the resistor increases, it is harder for the current to flow, so the current should go down.

If we have multiple resistors in parallel, the fraction of the current through one resistor is equal to the parallel combination of the other resistances, divided by the sum:

$$i_{R_x} = i_{total} \frac{(\sum_{n=1}^{N} \frac{1}{R_n} - \frac{1}{R_x})^{-1}}{(\sum_{n=1}^{N} \frac{1}{R_n})^{-1}}$$

Problem 3.4
Find the currents in each of the resistors in the following circuits.



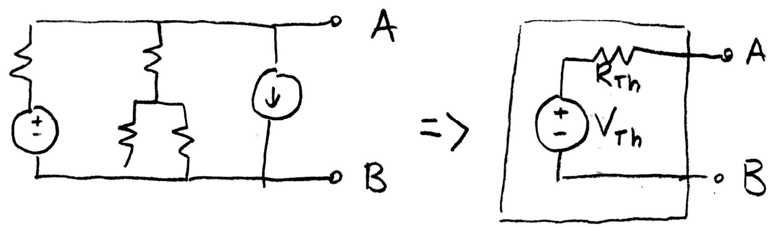
3.5 Equivalent Circuits

Honestly, most of the circuits that you will need to deal with when you are making stuff (and not doing homework problems) will be pretty simple, and won't be hard to analyze. Nodal analysis will always work, and you can make this even easier by reducing the complexity by using series parallel reductions and using current and voltage dividers. But occasionally you will find a circuit that is hard to analyze, or have a design problem where you will need to find a component that makes a voltage or current somewhere else in the circuit to the right value. Here it would be great if you could simplify the rest of the circuit to yield something simple like a voltage divider, when the relationship between the device you can adjust and the value you are interested in is easy to understand. This is especially true when there are some non-linear elements in the circuit, where understanding the region of operation of the non-linear device is important. Fortunately it is almost always possible to create these type of simplifications.

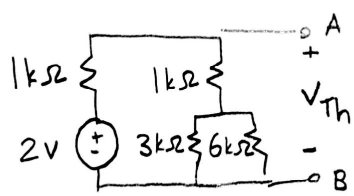
The basic idea behind these simplification is a concept we have used before: creating the iV curve for an electric device. In the previous chapter we created these curves for primitive devices like voltage sources, resistors and diodes, but we also showed how we could model more complicated device like a battery as a combination of a resistor and a voltage source, or a solar cell as a diode in parallel with a current source dependent on the light intensity. What we show in this section is that any combination of linear resistors, voltage sources and current sources can be modelled by a single resistor in series with a single voltage source. This single resistor, single voltage source model is called the *Thevenin Equivalent circuit*. The iV of any complex linear circuit can also be modeled by a current source in parallel with a resistor, and this model is called the *Norton Equivalent* circuit. The reason this is possible is simple. If all our devices are have a linear relationship between current and voltage (true for resistors, voltage and current sources) then the resulting relationship between current and voltage of any combinations of these devices will also be linear. This means if we take any two nodes in a linear circuit and pretend that all the components form a new device, see the figure below, this relationship between current and voltage will be a straight line in the i-V plane. This curve can be defined by a line, $V = i \cdot R + V_{th}$, where R is a resistance and V_{th} is a voltage source. This line can also be generated by a voltage source in series with a resistor.

3.5.1 Thevenin Equivalent

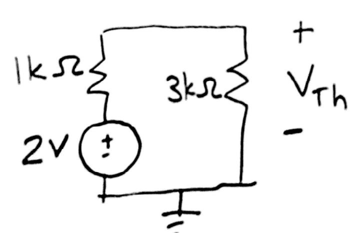
Thevenin's theorem states that a linear two-terminal circuit can be replaced by an equivalent circuit composed of a voltage source and series resistor, as shown below:



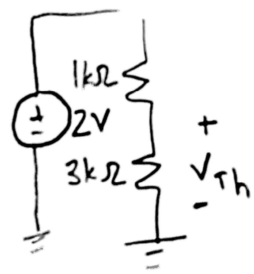
Further the theorem states that you can't make any measurement using just the nodes A and B that can tell the difference between the two circuits. this give us a hint of how to find the value of the voltage source and resistance for the equivalent circuit. One simple measurement we can make is to measure the voltage between A and B with nothing connected between the two terminals. In the Thevenin circuit, this voltage is obvious. Since there is no current running between A and B, the voltage drop across the resistor must be zero ($V = i \cdot R$), so the measured voltage is just V_{th}. Remember that since no current is flowing between the terminals, this is the "open-circuit" voltage. Since the open circuit voltage is equal to the value of the voltage source in the equivalent circuit, it is often called the Thevenin voltage. We can find this voltage by using nodal analysis on the actual circuit with no connection between A and B.



Let find the Thevenin equivalent voltage of the circuit shown above. We could use nodal analysis, and have two voltages to solve for, but in this case we can also solve the problem using series parallel reductions, so let's do it that way. We can immediately see that the 3kΩ and 6kΩ resistors are in parallel, so we can combine them into one resistance: $(\frac{1}{6k\Omega} + \frac{1}{3k\Omega})^{-1} = 2k\Omega$. This 2kΩ resistor is in series with the 1kΩ resistor above it, so we can add those together to form a single 3kΩ resistor. We end up with the circuit below:

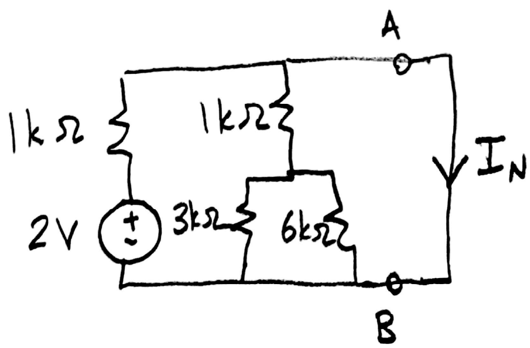


Here, we want to find the voltage across one of two resistors. This looks like a voltage divider. We can re-draw it to make that relationship clearer:

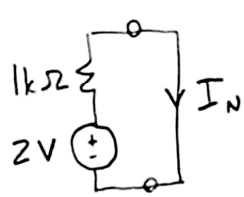


Now we can use our voltage divider equation, $V_{Th} = V_{in}\frac{3k\Omega}{3k\Omega+1k\Omega}$, to solve for V_{Th}. We end up with $V_{Th} = 1.5$V.

Now we need to make another measurement so we can figure out what the equivalent series resistor should be. Another easy measurement to make is the current that flows when we short together nodes A and B. This is the short circuit current.



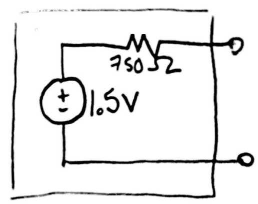
Remember the current through any resistor where both terminals of the resistor connect to the same node is zero (there is no voltage across the resistor so the current must be zero). This means that the $1\,\text{k}\Omega$, $3\,\text{k}\Omega$ and $6\,\text{k}\Omega$ resistors can be removed, since there is no voltage across them.



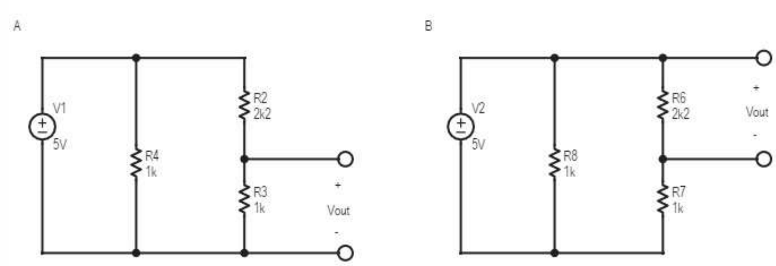
Finding the current in this case is easy, since we can just use Ohm's Law: $I_N = \frac{V}{R} = \frac{2V}{1k\Omega} = 2mA$.

The equivalent resistor must then be $1.5V/2\,\text{mA} = 750\,\Omega$

Now that we have the Thevenin voltage and resistance, we can plug them back in to our black box model to come up with the following Thevenin equivalent:



Problem 3.5 : Thevenin equivalent
Find the Thevenin equivalents of circuit A and circuit B below, at the ports indicated by V_{out}. Try to apply the rules you've learnt in this chapter.

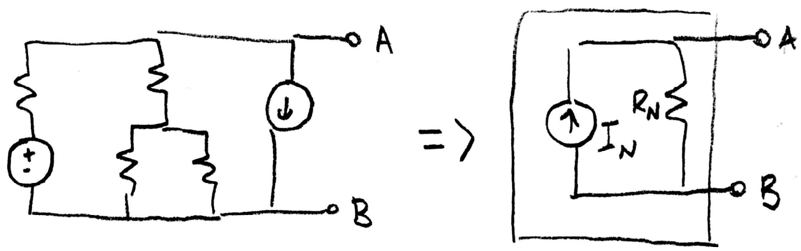


3.5.2 Norton Equivalent

It turns out that any linear i-V curve can also be represented by a current source in parallel with resistor, in addition to the Thevenin equivalent that we just described. Previously we said that any linear circuit could be represented by $V = i \cdot R_{Th} + V_{Th}$, where R_{Th} is the Thevenin equivalent resistance and V_{Th} is the Thevenin equivalent voltage. But dividing both sides of this equation by R, and rearranging terms gives:

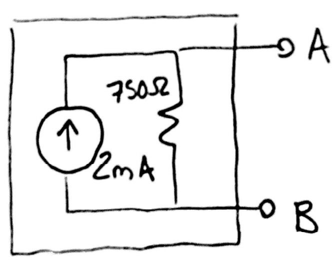
$$i = \frac{V}{R} + \frac{V_{Th}}{R_{Th}}$$

Since the current is the sum of two terms, it could be modeled by two elements in parallel. One is a resistor, since the current is proportional to a voltage, and the other term is a constant current, which can be modeled as a current source. Thus we get **Norton's theorem** which states that a linear two-terminal circuit can be replaced by an equivalent circuit composed of a current source and parallel resistor, as shown below:

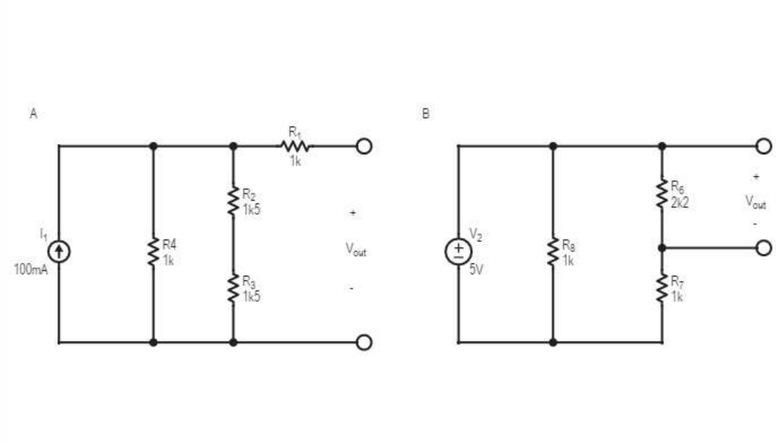


We solve this in a very similar manner to solving for Thevenin circuits. In this case the value of the current source is just the value of the short-circuit current, since with no voltage across the terminals, the current through the resistance must be zero. We found this before to be 2 mA.

Since the Norton resistance is equal to the Thevenin resistance, we use the same value we found before, 750 Ω.

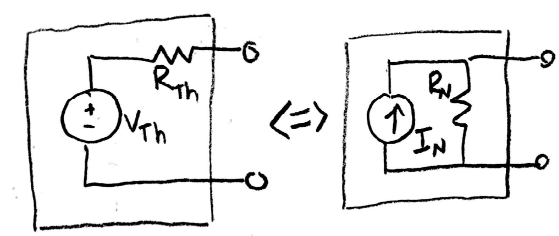


Problem 3.6 : Norton equivalents
Find the Norton equivalent of the following circuits.



3.5.3 Converting from Thevenin to Norton

As we saw above, the Thevenin and Norton resistances are equivalent. Thevenin voltage and Norton current can also be related using Ohm's Law:

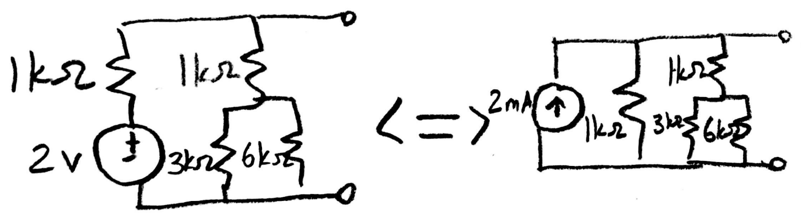


$$R_{Th} = R_N$$

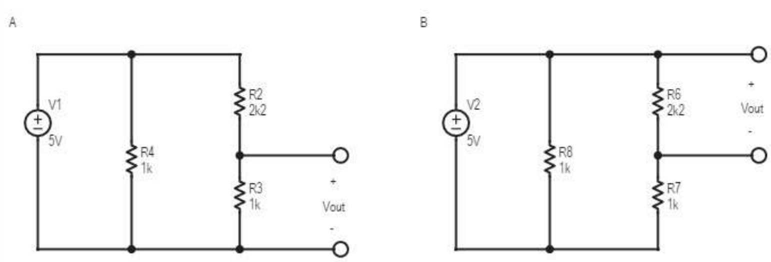
$$V_{Th} = I_N R_N$$

$$I_N = \frac{V_{Th}}{R_{Th}}$$

This result can be applied more generally to convert a voltage source in series with a resistor into a current source in parallel with a resistor, and vice versa. We can use this to simplify our nodal analysis. For example, if we wanted to work with a current source rather than a voltage source in the circuit we solved earlier, we could solve for $I_N = \frac{2V}{1k\Omega} = 2mA$ and update our circuit as shown below:



Problem 3.7
Converting from Thevenin to Norton:
Find the Norton equivalent of the following circuits - you derived their Thevenin equivalents earlier.
Hint: The Thevenin equivalent resistance of a circuit is equal to the Norton equivalent resistance of the circuit.



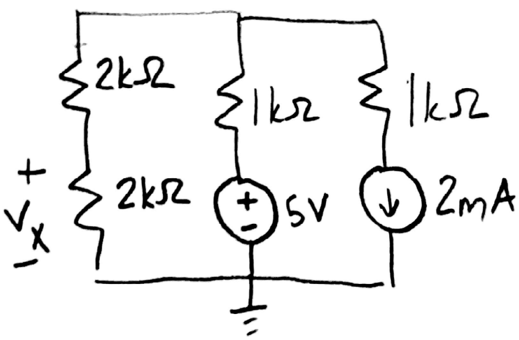
3.6 Superposition

For linear circuits there is one addition trick we can use to simplify our analysis to better understand how different sources (voltage, current) affect the circuit: we can take advantage of linearity. For linear systems, the voltage at any node, or the current through any device will vary linearly with any of the sources in the circuit. This means that the voltage at a node can be written as:

$$V_A = k_1 \cdot V_1 + k_2 \cdot V_2 ... + R_1 \cdot I_1 + ...$$

where V_i are the voltage sources in the circuit, I_i are the current sources in the circuit, and k_i and R_i are the proportionality constants. Notice that the voltage at each node is a simple sum of the contributions of the different sources. Since each contribution doesn't depend on the other sources, we can find the output voltage by computing the output from each source independently and then adding these contributions together. Basically, it means that we will get the same result by solving the entire equation at once or solving different sources separately and then adding the results together.

Let's try solving the circuit below. We can always use straight nodal analysis. For this circuit, we could solve for the voltage at the node at the top of the circuit - let's call it V_A. We could write our KCL equations as shown below:



$$\frac{V_A - 5V}{1k\Omega} + \frac{V_A}{4k\Omega} + 2mA = 0$$

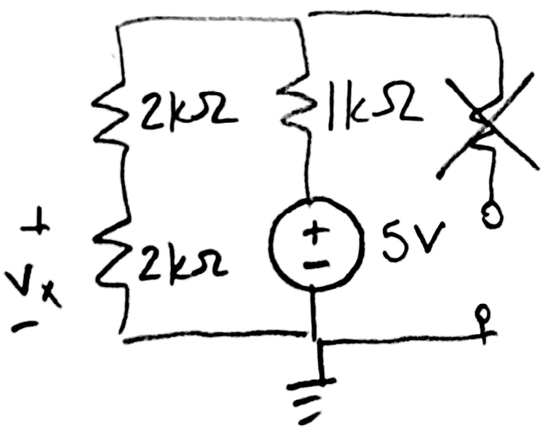
$$5V_A - 20V + 8V = 0$$

$$V_A = 2.4V$$

Then we could use a voltage divider on the left branch of the circuit:

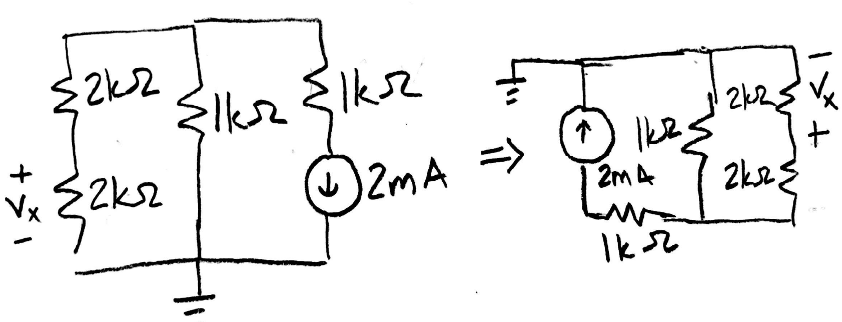
$$V_x = V_A \frac{2k\Omega}{2k\Omega + 2k\Omega} = \frac{V_A}{2} = 1.2V$$

However, since we have two independent sources, we can also use superposition to solve this circuit. Let's start by finding the contribution of V_x that comes from the voltage source. First, we'll zero out all other independent sources besides the voltage source of interest, which in this case just means shorting the 2mA current source to create the circuit shown below:



We'll ignore the resistor on the right side since it's floating and no current can flow through it. We can find Vx by using a voltage divider: $V_{x,v} = 5V * \frac{2k\Omega}{2k\Omega+2k\Omega+1k\Omega} = 2V$.

Now we can solve for the contribution from the current source. First, we'll zero out the voltage source by replacing it with a wire, yielding the circuit below:



The right-hand circuit is redrawn to show the circuit a bit more clearly. Note the polarity of V_x now that we've flipped around the circuit. There are multiple ways to solve this, but we'll use a current divider to find the current through the 2kΩ resistor, then multiply it by the resistance to find V_x. One branch in our current divider has a resistance of 1kΩ. The second branch has a resistance of 4kΩ, since we have two 2kΩ resistors in series. The 1kΩ resistor at the bottom of the circuit does not contribute to the current divider - since it is not in parallel with any other component, the entirety of the 2mA current flows through it.

$$I_r = 2mA * \frac{1k\Omega}{1k\Omega + 4k\Omega} = \frac{2mA}{5} = 0.4mA$$

We have to be careful with the polarity here. We've solved for a current that flows from the marked negative to the marked positive terminal of V_x. This will give us a negative voltage.

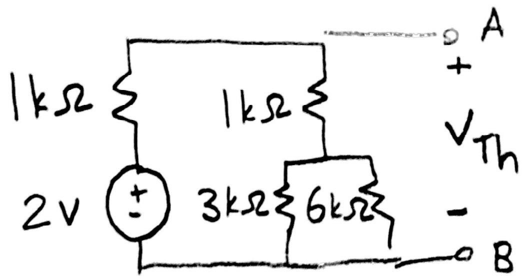
$$V_{x,i} = -0.4mA * 2k\Omega = -0.8V$$

Now that we have the contributions to V_x from both independent sources, we can simply add them together to find the total V_x:

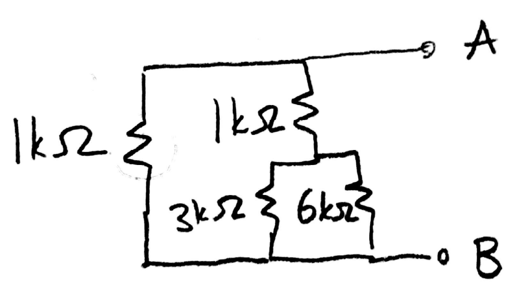
$$V_x = V_{x,v} + V_{x,i} = 2V + (-0.8V) = 1.2V$$

Note that this is the same answer that we got from nodal analysis.

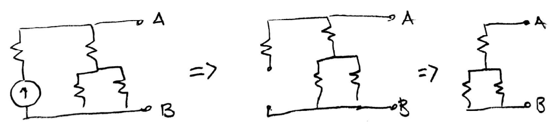
Superposition also provides another method to solve for the Thevenin/Norton equivalent resistance. Because of superposition the actual circuit and the equivalent should still be the same if we set all sources to zero. The Thevenin equivalent circuit becomes just a single resistor, and the real circuit becomes a collection of resistors. Going back to the circuit we worked on before:



To solve for the equivalent resistance between terminals A and B, we first need to set all independent sources to zero. We want zero voltage across the voltage sources, which means we need to short them (replace them with a wire). We want no current across the current source, which means we need to open them (take them out and leave the two nodes unconnected). In our circuit, we only have one voltage source, which we'll replace with a straight wire. Once that's done, our circuit looks like this:

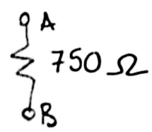


If we'd had a current source in place of the voltage source, we would cut it out, leaving the 1kΩ resistor unconnected to anything. A dangling resistor makes no difference in the circuit since current can only flow through a loop, so we would just ignore the resistor, as shown below:

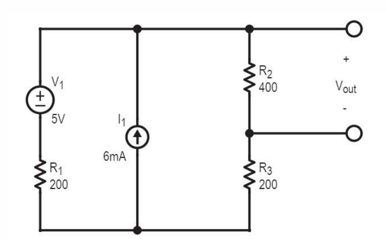


Let's return to our original circuit. We can simplify the right branch just as we did when finding the Thevenin voltage, giving us this circuit:

This is just two resistors in parallel. We can easily combine these resistors to form a single resistor, as shown below:



Problem 3.8
Superposition:
1. Find the output voltage of the following circuit using superposition.
2. Find the Thevenin equivalent of the following circuit. Your results from part 1 should help you with this.



3.7 Solutions to Practice Problems

Solution 3.1:
We can always choose one of the nodes in a circuit to be the "reference", or "ground" to simplify the problem. In this case we choose the bottom node for simplicity.
Then, there is only one voltage node of interest we need to solve for - the node connecting R1, R2 and R3 together. Let's call this node V_x.
By KCL, we know that $I1 + I2 + I3 = 0$. Then we can write the following questions and solve for V_x:

$$I1 + I2 + I3 = 0$$

$$\frac{V_x - 12}{1k} + \frac{V_x}{2.2k} + \frac{V_x - 24}{2.2k} = 0$$

$$2.2 \cdot V_x - 2.2(12) + V_x + V_x - 24 = 0$$

$$4.2v_x = 50.4$$

$$V_x = 12V$$

$$I1 = \frac{12 - 12}{1k} = 0$$

$$I2 = \frac{12}{2.2k} = 5.45mA$$

$$I3 = \frac{12 - 24}{2.2k} = -5.45mA$$

Solution 3.2:

$$R_{eq} = R7 + R6 \parallel (R5 + R4 \parallel (R3 + R1 \parallel (R2 + R8)))$$

Make sure you understand how this equation came about. It simply summarises the series and parallel relationships in the circuit.

$$R_{eq} = 1k + 1k \parallel (1k + 1k \parallel (1k + 1k \parallel (1k + 1k)))$$

$$R_{eq} = 1k + 1k \parallel (1k + 1k \parallel (1k + 1k \parallel 2k))$$

$$R_{eq} = 1k + 1k \parallel (1k + 1k \parallel (1k + \frac{2}{3}k))$$

$$R_{eq} = 1k + 1k \parallel (1k + 1k \parallel (1.667k))$$

$$R_{eq} = 1k + 1k \parallel 1.625k$$

$$R_{eq} = 1.619k$$

Solution 3.3:
Circuit A:

$$V_{out} = 12 \cdot \frac{R2}{R1 + R2}$$

$$V_{out} = 12 \cdot \frac{2.2k}{1k + 2.2k} = 8.25V$$

Circuit B:

$$V_{out} = 12 \cdot \frac{R3}{R3 + R4}$$

$$V_{out} = 12 \cdot \frac{1k}{1k + 2k2} = 3.75V$$

Solution 3.4:
Circuit A:

$$I1 = 100mA \cdot \frac{2k2}{1k + 2k2} = 68.75mA$$

$$I2 = 100mA \cdot \frac{1k}{1k + 2k2} = 31.25mA$$

Circuit B: The current divider equation doesn't scale nicely to more branches. So, when you see a circuit like this, it is easier to just use nodal analysis. First, find the voltage across all the resistors:

$$V = 100mA \cdot (1k \parallel 2k2 \parallel 4k7) = 100ma \cdot 600 = 60V$$

$$I3 = \frac{V}{1k} = 60mA$$

$$I4 = \frac{V}{2k2} = 27.27mA$$

$$I5 = \frac{V}{4k7} = 12.76mA$$

Solution 3.5:
Circuit A:

$$R_{TH} = (R4 + R2) \parallel R3 = 760\Omega$$

$$V_{TH} = 5 \cdot \frac{1k}{2k2 + 1k} = 1.56V$$

Circuit B:

$$R_{TH} = (R8 + R7) \parallel R6 = 1.05k\Omega$$

$$V_{TH} = 5 \cdot \frac{2k2}{2k2 + 1k} = 3.44V$$

Solution 3.6:
Circuit A:

$$R_N = R1 + R4 \parallel (R2 + R3) = 1.75k\Omega$$

To find I_N, we short the output ports and find the current flowing through them.
The voltage across the current source under those conditions are:

$$V_{current} = 100mA \cdot (R4 \parallel (R2 + R3) \parallel R1) = 100mA \cdot 429\Omega = 42.9V$$

$$I_N = \frac{V_{current}}{R1} = 42.8mA$$

Circuit B:

$$R_N = (R8 + R7) \parallel R6 = 1.05k\Omega$$

$$I_N = \frac{5V}{1k} = 5mA$$

Solution 3.7:
Circuit A:

$$R_N = R_{TH} = 760\Omega$$

$$I_N = \frac{5}{2.2k} = 2.27mA$$

Circuit B:

$$R_N = R_{TH} = 1.05k\Omega$$

$$I_N = \frac{5}{1k} = 5mA$$

Solution 3.8:
1.) The general procedure is as follows:
First calculate the contribution of V1 to the output voltage (call this V_{out1}), by removing the current source (replacing it with an open circuit).
Then, calculate the contribution of I1 to the output voltage (call this V_{out2}) by removing the voltage source (replacing it with a short circuit).
Finally, add the two voltages together to get V_{out}.

$$V_{out1} = 5V \cdot \frac{R2}{R1 + R3} = 2.5V$$

$$V_{out2} = V_{isource} \cdot \frac{R2}{R2 + R3} = 6mA \cdot R1 \parallel (R2 + R3) \cdot \frac{R2}{R2 + R3}$$

$$= 0.9V \cdot \frac{400}{600} = 0.6V$$

$$V_{out} = V_{out1} + V_{out2} = 3.1V$$

2.) We have already found V_{TH} in part 1 of the question - $V_{TH} = 3.1V$.

$$R_{TH} = R2 \parallel (R1 + R3) = 200\Omega$$