Chapter 6

Capacitors and RC Circuits

Up until now, we have analyzed circuits that do not change with time. In other words, these circuits have no dynamic elements. When the behavior of all elements is independent of time, the behavior of the circuit as a whole will be independent of time as well. This behavior is called the **steady-state** behavior of circuits. It can also be thought of as the equilibrium a circuit reaches after a sufficient period of time.

In this chapter, we will explore a new type of linear device called a **Capacitor**. These devices are used for storing energy in a circuit, which allows us to have memory, adjust the circuit's response to input voltages of different frequencies, understand the gate input of MOS transistors, and do many other useful things. Although capacitors do not impact the steady-state behavior, they will change the way in which voltages and current transition from one state to another.

6.1 Device Characteristics

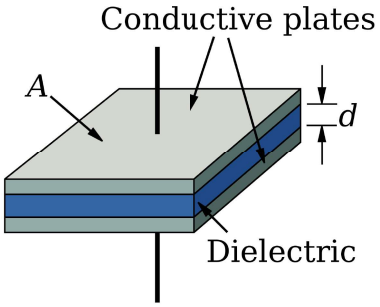


Figure 6.1: Capacitors store energy between two charged plates

Capacitors are two-terminal devices that store energy in an electric field between charged parallel plates, seen in Figure 6.1.[1] When we try to force a current where there is a gap, positive charge

[1] https://en.wikipedia.org/wiki/Capacitor

builds up on the positive terminal, and negative charge builds up on the negative terminal. Thus, there is a voltage (and, therefore, electric field) across this gap.

The **capacitance** of a capacitor is governed by the area of these plates, the distance between them, and the material, called the **dielectric**, between the two plates according to the following equation:

$$C = \frac{\epsilon A}{d}$$

where ϵ is the dielectric constant, A the area of the plates, and d the distance between them. For the purposes of this class, you do not need to worry about this equation; however, it shows that the capacitance of a capacitor is a function of its shape. The larger the area of the plates and smaller the gap, the higher the capacitance.

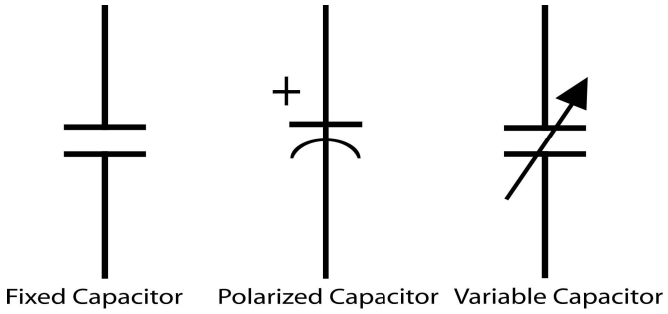


Figure 6.2: Electronic Symbols for Capacitors

Some capacitors are unpolarized, meaning the two terminals are interchangeable, like a resistor, and others are polarized, meaning they have a distinct anode and cathode like a diode. The cathode of the latter type is labeled with either a "-" or a stripe down the side, shown in Figure 6.3.[2] These also have different circuit diagrams, seen in Figure 6.2.[3] If a polarized capacitor is connected backwards, part of the internal insulator erodes, the capacitor becomes a short circuit, the huge amount of current it now conducts boils the electrolyte fluid, pressure builds up, and the capacitor explodes. In other words, double check the polarity before connecting these!

6.1.1 Governing Equations

We previously mentioned a capacitor is a **linear** element; however, it is not a resistor. How can this be? Capacitors actually relate stored charge Q in coulombs to the voltage across the capacitor V in volts:

$$Q = CV \tag{6.1}$$

As we saw before, C is the capacitance of the specific capacitor with which we are working, and it is a constant. Capacitance is measured in Farads (F). Typically we don't work with Q in our circuit calculations. Instead, we prefer working with circuits in terms of current i and voltage V.

[2] https://en.wikipedia.org/wiki/Electrolytic_capacitor
[3] See 1

Figure 6.3: Polarized Electrolytic Capacitors

We know current is the flow of charge Q over time, or dQ/dt, and we can exploit this to modify the above equation by taking the derivative of both sides with respect to time:

$$\begin{aligned} Q &= CV \\ \frac{dQ}{dt} &= C\frac{dV}{dt} \\ i &= C\frac{dV}{dt} \end{aligned}$$

This equation, which relates the current through the capacitor to the change in voltage across it, is much better to work with. Note that the equation has a linear relationship between i and V. If you double i (for all time), V doubles and vice versa.[4] EE 102A will formally define linear systems.

Because we have dV/dt in this equation, we can deduce it takes current to change a voltage across a capacitor, and the faster you try to change the voltage, the more current that is required. Since most circuits can supply only a fixed amount of current, **capacitors will prevent the voltage across them from changing rapidly**. Said differently, capacitors look like voltages sources for short periods of time (for a small dt). While the voltage across a capacitor can't change rapidly, there are no constraints on how fast the current can change. The current through a capacitor can change from 0 to a large value instantly: current can change abruptly.

If you think of charge as a fluid, then you can think of a capacitor like a large tank.[5] The height of the water in the tank represents the voltage on the capacitor. While we can instantaneously turn the flow of water off and on (changing the current), we cannot instantaneously change the water level of this tank. Similarly, we cannot instantaneously change the voltage across the capacitor. Its

[4] Said differently, d/dt might be a strange operator, but it is linear

[5] You need to be careful with this analogy, since there is plus and minus charge, and there is not negative water. When water flows into a tank to fill it up, it only flows into the top of the tank. With a capacitor there are really two tanks. Charge flows into the top tank, which starts filling it up, but and equal amount of charge also leaves the bottom tank, filling the bottom tank with negative charge. A better fluid analogy would be to model a capacitor as two tanks. The first one is right side up and empty, and the second is upside down and full. When liquid flows into the top tank, to start to fill it up, and equal amount of liquid flows out of the bottom tank, and is replaced by bubbles.

value changes as a result of the integration of the current being added to the tank/capacitor, and the value of the capacitance is related to the area of the tank.

Because capacitors are energy storage devices, we are also interested in determining the energy stored in a capacitor. We know that power describes energy at a specific instant of time. Therefore, we can solve for the power P, substitute values from the equations above, and integrate over time to get energy:

$$\begin{aligned} E &= \int P dt \\ &= \int iV dt \\ &= \int C\frac{dV}{dt}V dt \\ &= \int_0^{V_{final}} CV dV \\ E &= \frac{1}{2}CV_{final}^2 \end{aligned}$$

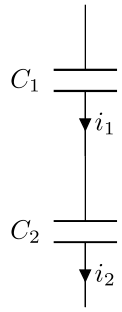
This equation tells us that the voltage across a capacitor determines the energy it stores.

6.2 Capacitors in Series and Parallel

We would like to create equations to simplify multiple capacitors in a circuit. To do this it would be nice to see if there was an equivalent "resistance" for a capacitor. We can't really find one, since the current depends on ΔV not V, but we can say that $\Delta V = i\frac{\Delta T}{C}$. This is interesting in two ways. First is says that the effective resistance is related to $1/C$, so circuits where R adds will need to combine $1/C$. Second it shows how the resistance depends on the ΔT. If ΔT is small, the effective resistance will be small, but if the time is large, the resistance will also be large.

6.2.1 Capacitors in Series

Like with resistors, when we have multiple capacitors in a circuit, we often want to replace them with one equivalent capacitor. We'll start by analyzing capacitors in series:



Using our equation for the current through a capacitor, we have $i_1 = C_1\frac{dV_1}{dt}$ and $i_2 = C_2\frac{dV_2}{dt}$ where V_1 and V_2 are the voltages across C_1 and C_2 respectively. We want to find one capacitor with

capacitance C_{eq} such that the voltage across it is $V_1 + V_2 = V_{tot}$ and the current through it is $i = i_1 = i_2$. We know from KCL that $i_1 = i_2$. With this information, we now rearrange the equations to get:

$$i_1 = C_1 \frac{dV_1}{dt} \qquad i_2 = C_2 \frac{dV_2}{dt}$$
$$\frac{i_1}{C_1} = \frac{dV_1}{dt} \qquad \frac{i_2}{C_2} = \frac{dV_2}{dt}$$

$$i\left(\frac{1}{C_1} + \frac{1}{C_2}\right) = \frac{dV_{tot}}{dt}$$

$$i = \left(\frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}\right) \frac{dV_{tot}}{dt}$$

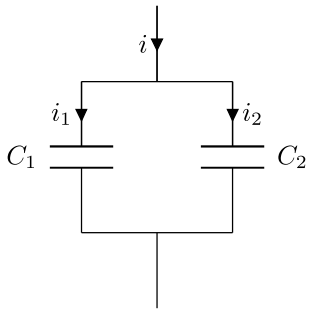
This looks a lot like our equation for current through a capacitor. Thus, we have the following relation for capacitors in series:

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n}$$

This makes sense intuitively, as both capacitors will be experiencing the same current, and the voltage across both capacitors will increase with respect to this current. Thus, the total voltage across both capacitors will increase at a greater rate than either of the voltages accross individual capacitors. We also notice that this relation looks like the relation for resistors in parallel.

6.2.2 Capacitors in Parallel

We also want to be able to replace capacitors in parallel. Let's analyze the following:



By KVL, we know that the voltage across C_1 must equal the voltage across C_2. Call this voltage V. Also, by KCL, we know that $i = i_1 + i_2$. We substitute the currents through each capacitor:

$$\begin{aligned} i &= i_1 + i_2 \\ &= C_1 \frac{dV}{dt} + C_2 \frac{dV}{dt} \\ &= (C_1 + C_2) \frac{dV}{dt} \end{aligned}$$

Thus, we have the following relation for capacitors in parallel:

$$C_{eq} = C_1 + C_2 + ... + C_n$$

Going back to our water tank analogy, the summation makes sense. Putting two capacitors in parallel is like putting two water tanks in parallel. We also notice that this relation is similar to resistors in series. In this section, we have seen that the equivalent capacitance equations are the same as the equivalent resistance ones, but the series and parallel behaviors are flipped.

6.3 Capacitors in real circuits

Capacitors, like many electrical components, come in a variety of shapes and sizes. Larger capacitance and higher voltage compliance generally comes with larger size. Electrolytic capacitors have the largest capacitors per unit volume, but they have limited voltage and as polarized, as described before.

6.3.1 Capacitors to Control Supply Voltage

In our circuits, we often want a constant, DC voltage source. While this is perfectly fine on paper, in practice, we know that sources are not ideal. As you draw more current, especially if the change is sudden, voltage will drop. We can mitigate these effects by connecting a capacitor between Vdd and Gnd. We know capacitors resist changes in voltage, so the capacitor will work to keep Vdd constant. Essentially, the capacitor acts as a energy reserve for the circuit, supplying energy when the demands of the circuit exceed the battery's (or other voltage source's) capabilities.

6.3.2 Capacitors in MOS Transistors

In a MOS transistor, there is a capacitor between the gate and the source. From the equation of a capacitor

$$i = C \frac{dV}{dt}$$

we see that if $i = 0$, then $\frac{dV}{dt} = 0$. As a result, if there is no current, the voltage across the capacitor (from the gate to the source) remains constant. For example, if we simply disconnect the gate terminal from Vdd, without driving it to another voltage, it will remain at Vdd. Thus, we need a current to cause a change in voltage and change the gate voltage of the MOS transistor.

6.3.3 Real Wires

All real wires also have capacitance. As we saw above, this means wires require some charge to change their voltage. Remember that voltage is defined as potential energy per charge, so this observation makes sense in the context of our definition. This means that any time we want to change the voltage of a wire, we need charge to flow into it. The amount of charge we need to produce a given change of voltage in a certain amount of time is governed by the equation:

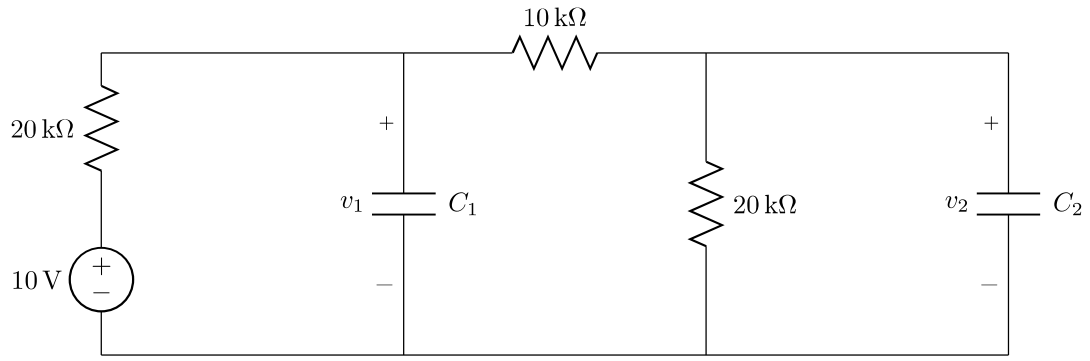
$$i = C\frac{dV}{dt}.$$

We're starting to see a theme. Capacitance governs how fast we can change voltages and the energy we need to do so. This result governs the speed and power consumption of modern electronics, including your computers!

6.4 RC Circuits

6.4.1 RC Circuits at DC

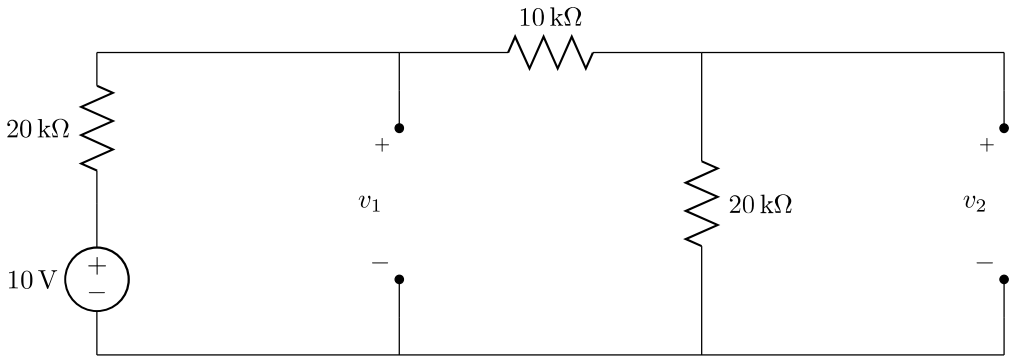
Now that we've seen how capacitors behave, we can use them in circuits. First we'll consider the case when capacitors are in circuits with DC sources. Let's examine the circuit below:



We want to find v_1 and v_2. First, notice that all voltage sources in this circuit are at DC, in other words, they output a constant voltage across them. This means there is no change in voltage over time. Refer back to the equation relating voltage and current across a capacitor:

$$i = C\frac{dV}{dt}.$$

We see that if $dV/dt = 0$, then $i = 0$ as well. In terms of our water tank analogy, the tanks are full, so no flow (current) is going to go into them. Thus, capacitors are DC open circuits. We can essentially remove them from the diagram before doing our analysis, shown below:



Now we can use the techniques we already know to deal with circuits with only resistors. In this case, both voltages can be found with voltage dividers:

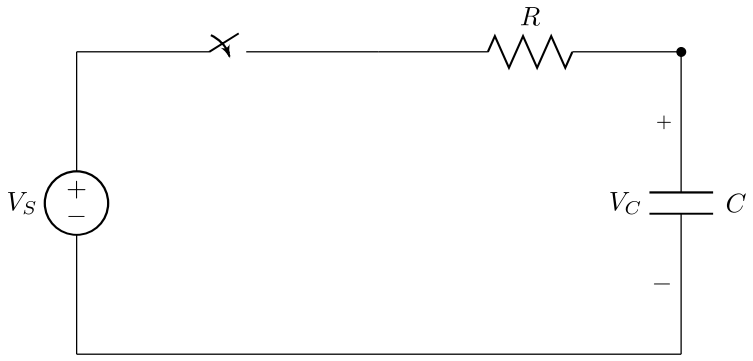
$$v_1 = 10\text{V}\left(\frac{20k\Omega + 10k\Omega}{20k\Omega + 20k\Omega + 10k\Omega}\right) = 6\text{V}$$

$$v_2 = 10\text{V}\left(\frac{20k\Omega}{20k\Omega + 20k\Omega + 10k\Omega}\right) = 4\text{V}$$

We find $v_1 = 6\text{V}$ and $v_2 = 4\text{V}$.

6.4.2 Charging a Capacitor

Often we will want to find out how a capacitor charges or discharges and the time it takes to do so. We'll first consider the former case. We know that the voltage across a capacitor cannot change instantaneously because the current cannot be infinite. This is equivalent to filling up the water tank with a finite flow of water into the tank. Let's look at the following circuit with DC voltage source V_S. Suppose that the switch was initially disconnected and then connects at time $t = 0$ and the initial value of V_C at $t = 0$ is 0V.



Let's look at the marked node between the resistor and the capacitor. We can apply KCL here:

$$\frac{V_S - V_C}{R} = i_C$$

$$\frac{V_S - V_C}{R} = C\frac{dV_C}{dt}$$

$$\frac{dt}{RC} = \frac{dV_C}{V_S - V_C}$$

$$\int_0^t \frac{1}{RC}dt = \int_0^t \frac{1}{V_S - V_C}dV_C$$

$$\frac{t}{RC} = -ln(V_S - V_C) + C_1$$

$$C_2 e^{\frac{-t}{RC}} = V_S - V_C$$

We can solve for C_2 by plugging in the initial condition $V_C = 0$ at $t = 0$, and we find that $C_2 = V_S$. Thus,

$$V_C(t) = V_S(1 - e^{\frac{-t}{RC}})$$

Now that we found the equation that describes charging a capacitor, we do not have to solve the differential equation every time we encounter an RC circuit. Note that the capacitor charges to match the voltage of V_S. This makes sense with our above model of capacitors at DC. If we wait a sufficiently long time, $V_C \to V_S$ and $i_C \to 0$. Often we replace RC with τ ("tau"). This is called the **time constant** of this RC circuit and has units of seconds. The fact that ohms and farads multiply to give us seconds probably seems a little weird at first. See if you can convince yourself of this by replacing Farads with coulombs per volt and ohms with volts per ampere. Then recall that current is the flow of charge, so amperes are coulombs per second:

$$\Omega \cdot F = \frac{V}{A} \cdot \frac{C}{V} = \frac{V}{A} \cdot \frac{A \cdot s}{V} = s$$

Replacing RC with τ, we have the charging equation:

$$V_C(t) = V_S(1 - e^{\frac{-t}{\tau}})$$

where $V_C(0) = 0$ and $V_C(\infty) = V_S$

The charging equation is illustrated in Figure 6.4.[6]. Notice that the capacitor reaches 95% of its final value after three time constants ($t = 3\tau$). Often, in practice, we will have events that trigger after a certain voltage threshold has been reached. We know all real wires have some amount

[6] https://en.wikipedia.org/wiki/RC_circuit

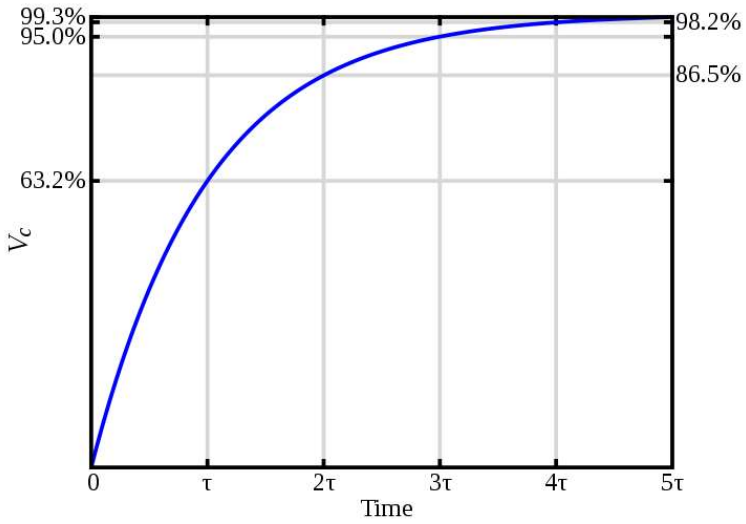
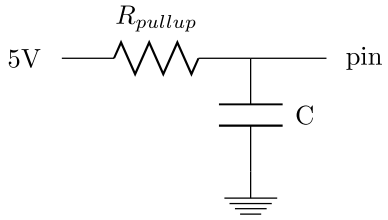


Figure 6.4: Charging a Capacitor

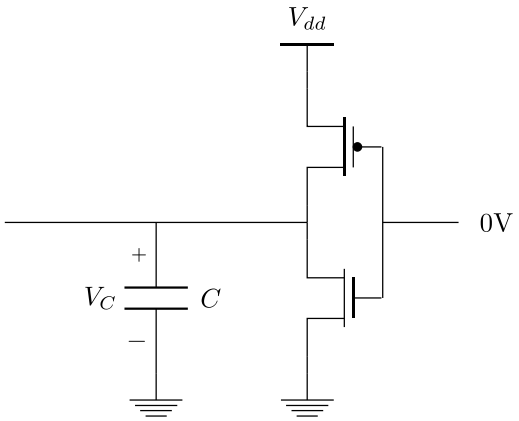
of capacitance (which can be modeled with a capacitor) and some small amount of resistance; therefore, we can now calculate how long it takes to drive a wire to a voltage we desire.



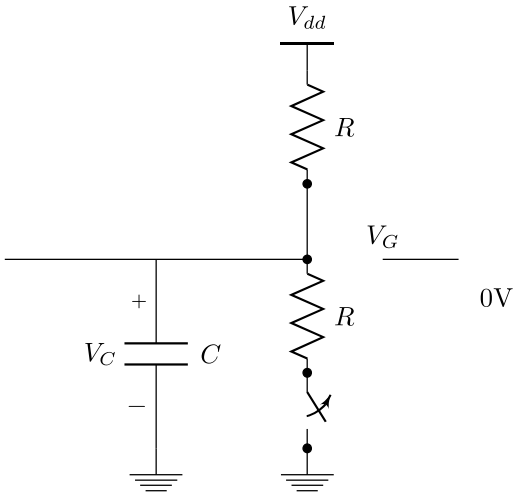
For example, on the useless box, the input switch pins were driven to 5V when the switches were disconnected. We now are equipped with the tools to find out exactly how much time elapsed between the switch opening and the pin reading HIGH. By modeling the circuit as an RC circuit, we can calculate the time to reach the threshold above which digitalRead() returns HIGH. Problems like these arise in many situations, from calculating the speed of communication protocols (for example, how fast your laptop and Arduino can communicate) to calculating the speed logic gates can turn on and off. We will examine the latter in the next section.

6.4.3 Discharging a Capacitor

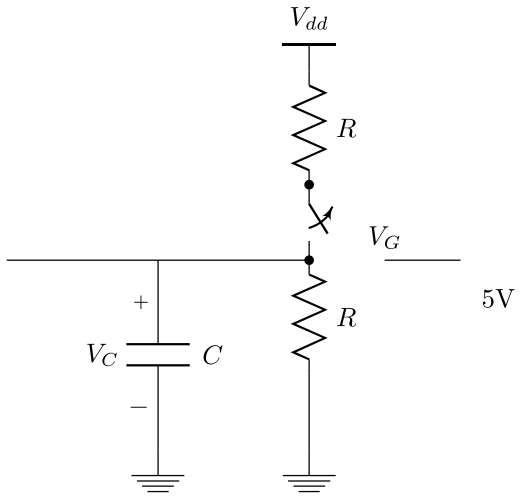
We motivate the discharging capacitor calculation with an example. We know computers are made of CMOS logic circuits. Let's examine the inverter below. This logic cell drives a wire, which we know has a small amount of capacitance.



We can model the transistors as voltage dependent switches. Since we are driving the input to 0V, the pMOS transistor will be on and the nMOS transistor will be off.



Assuming this circuit has been in this state for a sufficiently long time, we know that $V_C = V_{dd}$. At $t = 0$, let drive the gate voltage V_G to V_{dd} so that the pMOS transistor turns off and the nMOS transistor turns on:



Intuitively, we know that the capacitor now has stored energy in it, which will discharge across the resistor. We can characterize this over time by applying KCL at the node above the capacitor:

$$\frac{0 - V_C}{R} = C\frac{dV_C}{dt}$$

$$\frac{dt}{RC} = \frac{dV_C}{-V_C}$$

$$\int_0^t \frac{1}{CR}dt = \int_0^t \frac{1}{-V_C}dV_C$$

$$K_2 e^{\frac{-t}{RC}} = V_C(t)$$

We know that the initial value of $V_C(t)$ is V_{dd}, so we have:

$$V_C(t) = V_{dd}e^{\frac{-t}{RC}}$$

From this equation, we observe that the output wire of this logic gate cannot change from V_{dd} to Gnd immediately. In fact, this reality is what governs the speed of your computers, as all the digital logic in your CPU takes non-negligible time to switch states.

6.4.4 General form for charging and discharging

For circuits like those we saw above, we do not have to go through the process of solving a differential equation each time. The general solution is:

$$V_C(t) = A + Be^{\frac{-t}{\tau}}$$

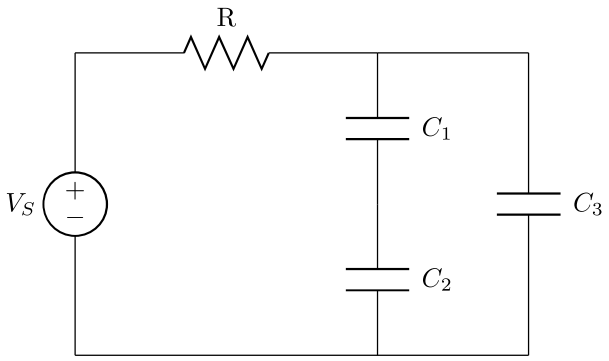
Here, $A + B$ is the initial value of V_C and A is the final value of V_C. Remember that $\tau = RC$.

6.4.5 Dealing with multiple capacitors, resistors, or sources

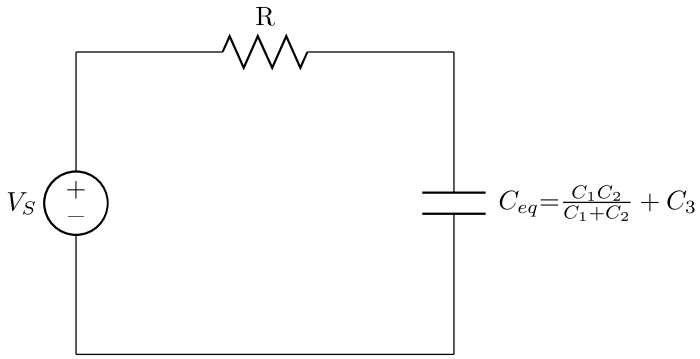
So far all the examples we've seen have only had one capacitor, one source, and one resistor. However, we can use the tools we have to simplify more complex circuits into this form.

Multiple Capacitors in a Circuit

Let's first consider what happens when we have multiple capacitors in a circuit:

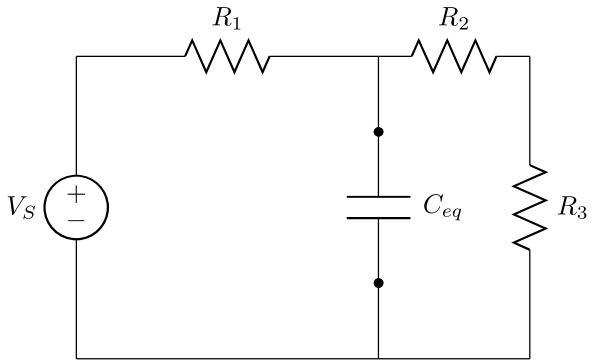


We can use the formulas for capacitors in series and parallel to collapse these capacitors into one capacitor with some C_{eq}. First collapse the two capacitors in series into $C_{eq1} = \frac{C_1 C_2}{C_1 + C_2}$. Next, we add capacitors in parallel to get $C_{eq} = \frac{C_1 C_2}{C_1 + C_2} + C_3$. Now we can redraw the circuit in a more familiar form:

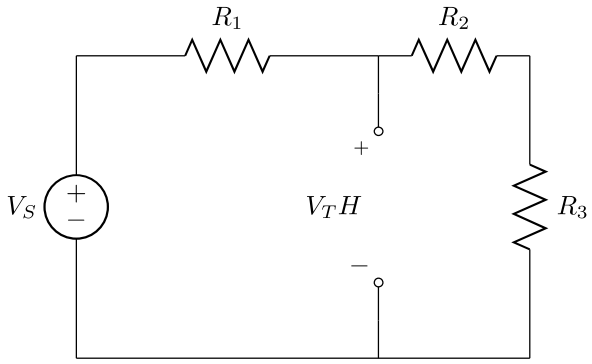


Multiple Resistors in a Circuit

Now let's consider a circuit with multiple resistors around some capacitor:



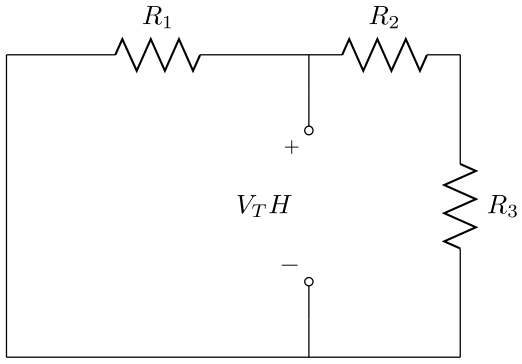
We know that a two terminal circuit with linear elements can be modeled by a circuit with a single voltage source and resistor in series, also know as a Thevenin equivalent circuit. Because we like working with RC circuits that only have one resistor and one source, we will try to collapse everything around the capacitor here into a Thevenin equivalent circuit:



First, we see that V_{TH}, the open circuit voltage across the terminals, can be found with a voltage divider:

$$V_{TH} = V_S \left(\frac{R_2 + R_3}{R_1 + R_2 + R_3} \right)$$

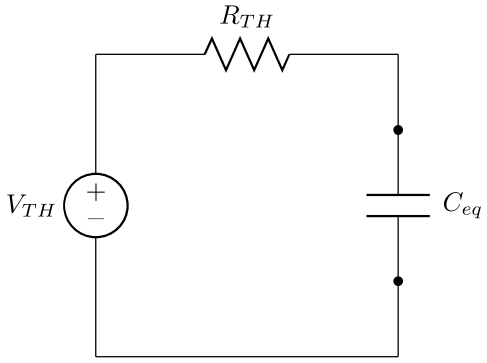
To calculate the Thevenin equivalent resistance R_{TH}, we set the voltage source to zero :



We see that R_2 and R_3 are in series with each other, and $(R_2 + R_3)$ is in parallel with R_1.

$$R_{TH} = (R_1||(R_2 + R_3)) = \frac{(R_2 + R_3)R_1}{(R_2 + R_3) + R_1}$$

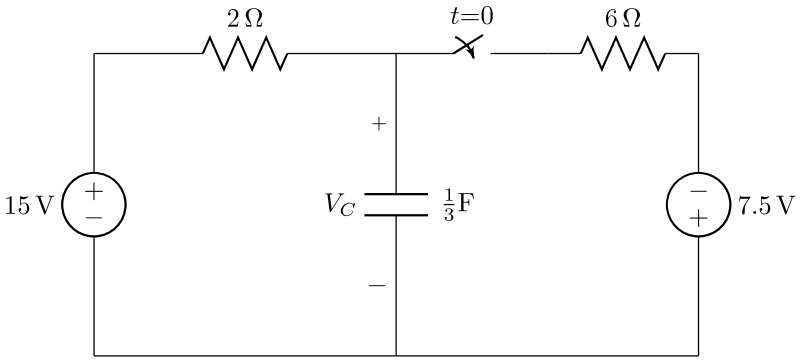
We now have a Thevenin equivalent circuit that connects to the terminals of the the capacitor:



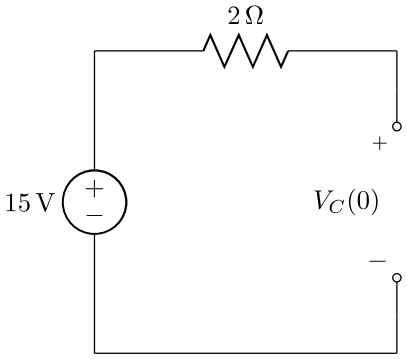
This circuit is now in a form we know how to solve. Multiple sources and multiple resistors are both handled using Thevenin equivalent circuits (they actually do make our lives easier!). In the numerical example below, we examine a circuit with multiple sources and multiple resistors.

6.4.6 A numerical example

Let's find $V_C(t)$ for $t > 0$ in the circuit below.[7] Assume the switch has been open for a long time and is closed at $t = 0$.

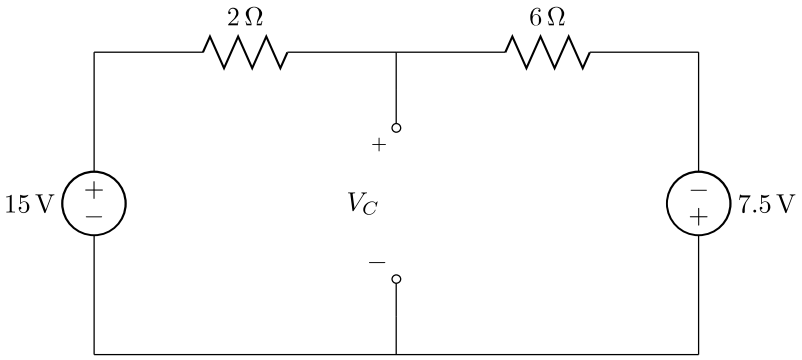


First we determine the initial voltage across the capacitor, $V_C(0)$. We know that the switch was open "for a long time." Thus, we can assume that the capacitor was acting as a short circuit before the switch is closed.

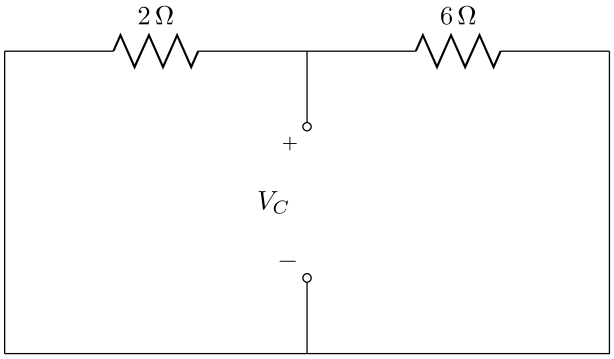


We see that $V_C(0) = 15V$. Now, we will consider what happens for $t > 0$. First, we wish to manipulate our circuit to get the problem into a form we know. (Alternatively, you can write out and solve the differential equations.) We will take out the capacitor and turn the remaining circuit into a Thevenin equivalent circuit. After we have the problem in a form with one source and one resistor, we can apply the general form equation.

[7] Taken from Fundamentals of Electric Circuits, Fifth Edition by Alexander, Sadiku



To find the Thevenin equivalent resistance, we zero all sources, leaving two resistors in parallel:



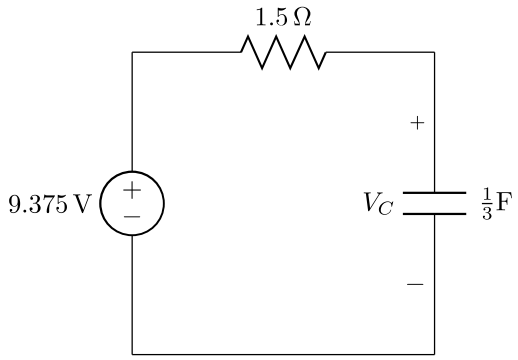
$$R_{TH} = \frac{2\Omega * 6\Omega}{2\Omega + 6\Omega} = 1.5\Omega$$

We also can see that the voltage drop is[8]

$$V_{TH} = 15V - 2\Omega(i) = 15V - 2\Omega\left(\frac{15V + 7.5V}{8\Omega}\right) = 15V - \frac{45V}{8\Omega} = 9.375V$$

Thus, our circuit for $t > 0$ is below:

[8]For a more detailed explanation of calculating Thevenin equivalent circuits or voltages and resistances, please refer to the respective sections of the reader.



Now we have a form we can solve easily with our general form equation.

$$V_C(t) = A + Be^{\frac{-t}{\tau}}$$

We know that:

- $V_C(0) = 15V$
- $R = 1.5\Omega$
- $V_C(\infty) = 9.375V$

We calculate:

- $A = V_{C_{final}} = V_C(\infty) = 9.375V$
- $A + B = 15 \rightarrow B = 5.625V$
- $\tau = RC = 1.5\Omega\left(\frac{1}{3}F\right) = \frac{1}{2}s$

And finally we substitute these values into our general form equation:

$$V_C(t) = (9.375 + 5.625e^{-2t})V \text{ for all } t > 0$$

6.5 Summary

- Capacitors are linear devices that store energy in electric fields. They can be polarized or nonpolarized.
- Voltage acrross a capacitor and current through it are related by $i = C\frac{dV}{dt}$
- The energy stored in a capacitor is $\frac{1}{2}CV^2$
- Capacitors in series can be replaced by an equivalent capacitor $\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n}$
- Capacitors in parallel can be replaced by an equivalent capacitor $C_{eq} = C_1 + C_2 + ... + C_n$
- In DC circuits, meaning all the voltage sources are constant, capacitors act as open circuits.
- The time constant of a capacitor is $\tau = RC$. Its units are seconds.
- The general equation for charging or discharging a capacitor in circuits with one voltage source (if charging) and one resistor is $V_C(t) = A + Be^{\frac{-t}{\tau}}$ where $A + B$ is the initial value of V_C and A is the final value of V_C.
- When you have multiple capacitors in a circuit, you can often combine them into a capacitor with some C_{eq}
- To deal with more complex problems, use a Thevenin equivalent circuit to change the problem into a form you know