Chapter 7

Impedance and Filters

A Way To Analyze RC Circuits

For this section, we'll assume you're comfortable with the following:

- the idea that any signal (a voltage or other quantity that varies with time) can be represented as a sum of sine waves
- the voltage/current relationship for capacitors ($i = C\frac{dv}{dt}$) and inductors ($V = L\frac{di}{dt}$)
- how RC and LR circuits behave for step inputs (e.g., when a switch closes, instantaneously changing the voltage)

Our goal in this section is to find a way to predict the behavior of an RC (resistor-capacitor) or RL (resistor-inductor) circuit in response to any input signal, not just to step inputs. This is a difficult problem, because inductors and capacitors cause integral and derivatives in the circuit equations, and things get really messy, really fast.

We're going to work around this by making two observations:

- Every signal can be represented as a sum of sine waves.
- Calculating what happens to a capacitor or inductor with a sinusoidal voltage or current is easy. The derivative and integral of a sine wave is just a cosine, which is the same as a sine but shifted left or right.

If we can describe what will happen to a sine wave of any frequency, then we will be able to predict what will happen to any possible signal. When you finish this section, you should be able to:

- Understand what a Bode plot is and how to use it
- Understand how to describe signals in decibels (dB), where is it commonly used and why we use it.
- Describe the relationship between voltage and current using impedance

7.1 Gain (and dB)

In this and later sections, we're going to talk a lot about the idea of "gain", the relative increase or decrease in signal magnitude. In the future, we'll be constructing amplifiers and other things that modify signals, and gain is an important metric for describing how they behave. The gain of a signal is simply the output signal magnitude divided by the input signal magnitude. A gain larger than 1 means that the signal was amplified (i.e., it came out larger); a gain less than one means it was attenuated (came out smaller).

It turns out that a log scale is more convenient for talking about gain, so we define the unit "bel", which is a 10X increase in *power*. For unknown reasons, EE's prefer to work in tenths of a bel, which are "decibels" and abbreviated as dB:

$$\text{gain in dB} = 10 \cdot \log_{10} \frac{\text{power out}}{\text{power in}}$$

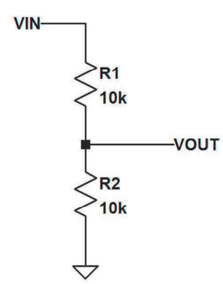
More often, we're measuring voltage rather than power. Since $P = I \cdot V$ and for a resistor, $I = \frac{V}{R}$, we can also express the dB gain in terms of voltage:

$$P_{in} = \frac{V_{in}^2}{R}$$

$$P_{out} = \frac{V_{out}^2}{R}$$

$$\text{gain in dB} = 10 \cdot \log_{10} \frac{V_{out}^2}{V_{in}^2} = 20 \cdot \log_{10} \frac{V_{out}}{V_{in}}$$

Example: Finding the gain of a voltage divider



$$Gain_{db} = \frac{V_{out}}{V_{in}} = 20 \cdot \log\left(\frac{10k}{10k+10k}\right) = 20 \cdot \log 0.5 = -6dB$$

Interesting observation: notice that the gain in dB is negative. This is because the gain is 0.5, which is less than 1.

Question: Can the gain in dB of a resistive divider ever be positive? [1]

[1] The gain in dB of a resistive divider can never be positive because the gain is always less than 1, ie. the output is always some fraction of the input.

7.2 Bode plots

A Bode plot is a very useful tool for expressing gain when the circuit comprises frequency dependent components such as capacitors and inductors. It simply plots the gain versus the frequency. An example of a Bode plot for a low pass filter is shown in Figure 7.1. This is called a low-pass filter because it provides the most gain at low frequencies, while providing less and less gain at higher frequencies.

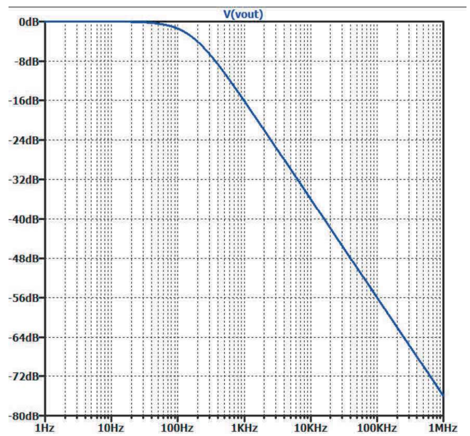


Figure 7.1: Bode plot of a low-pass filter

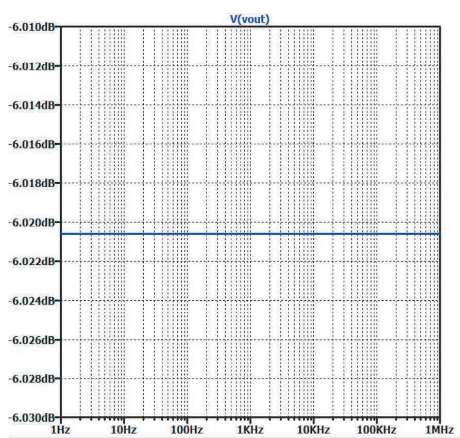


Figure 7.2: Bode plot of the resistive divider

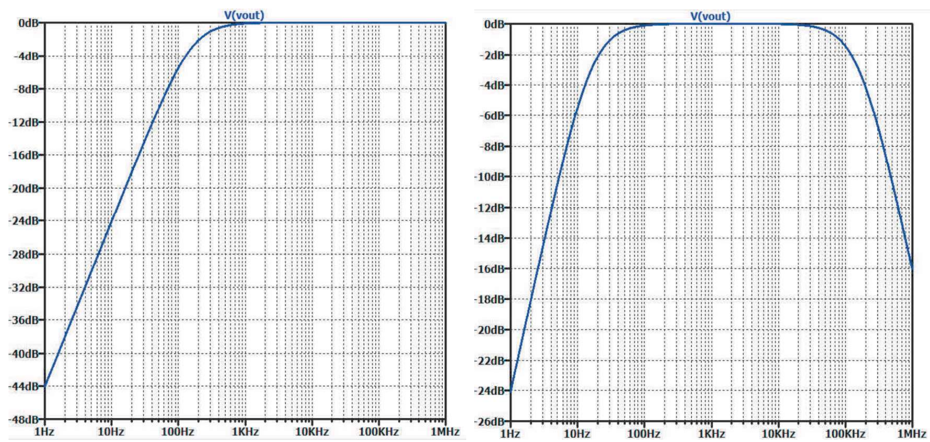
What makes a Bode plot a Bode plot is that both the gain and freq axis are plotted using logarithmic scales.The Y-axis is $Gain_{dB}$ is the gain measured in dB. And while this is plotted on a linear scale, dB is a logarithmic measure, so gain is plotted on a log scale. The X-axis, frequency, is explicitly plotted on a logarithmic scale which results in a log-log plot.

One reason for plotting in this way is because human hearing actually works on a logarithmic scale, which is why you'll often find the unit dB on the specifications for your audio devices. Another is simply that is makes for nice, clean plots from which useful information can be easily extracted, as you will see in later sections.

A Bode plot of a resistive network is relatively boring, since resistance is constant over frequency. Figure 7.7 shows the Bode plot of the resistive divider we just found the gain for. As you can see, it is simply a flat line at our calculated value of -6dB.

However, when capacitors and inductors are introduced to the circuit, these Bode plots become very useful. By using combinations of R, L, and C, we can create filters, and the Bode plots become much more interesting.

Filters remove unwanted frequencies from electrical signals. As we know, all signals can be decomposed into sinusoids of varying frequencies, and by removing the unwanted frequencies we can achieve a much cleaner signal. This is what equalizers do, by using filters. Figure 7.3 show the Bode plots of two other types of filters - high-pass filters and band-pass filters.



(a) Bode plot of a high pass filter (b) Bode plot of a bandpass filter

Figure 7.3: Bode plots of filters

Once we understand how to characterise the impedances of capacitors and inductors, which is discussed in the next section, we can use them to create these filters.

7.3 Generalized resistance (Impedance)

In previous chapters, we were able to use resistance and Ohm's Law, V= I R, to solve for the voltages and currents in many circuits. Now that we have added capacitors and inductors, we can no longer use Ohm's Law for these components. Wouldn't it be nice, if we could find some effective resistance for these new devices so we can use what we already know to solve circuits that have capacitor and inductors in them? Fortunately this is possible, and this section will show you how to do it. This generalization of resistance is called **impedance**, and is represented by '**Z**'.

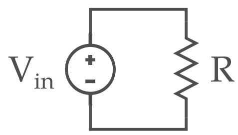
Resistance was defined as the ratio between voltage and current. Since the current through a capacitors and inductors depends on the rate of change of the signal, we can't define this ratio for an arbitrary input, but we can define it when the voltage across the device is sinusoidal. Thus we're going to observe the current when we put a sinusoidal voltage signal across the device. The voltage across the device is a function of time, given by the equation

$$V_{in} = \sin(2\pi f \cdot t)$$

Here, f is the frequency of the signal in Hz, and t is time.

7.3.1 Resistors

Let's start with a really simple circuit, with just the voltage source and a resistor:

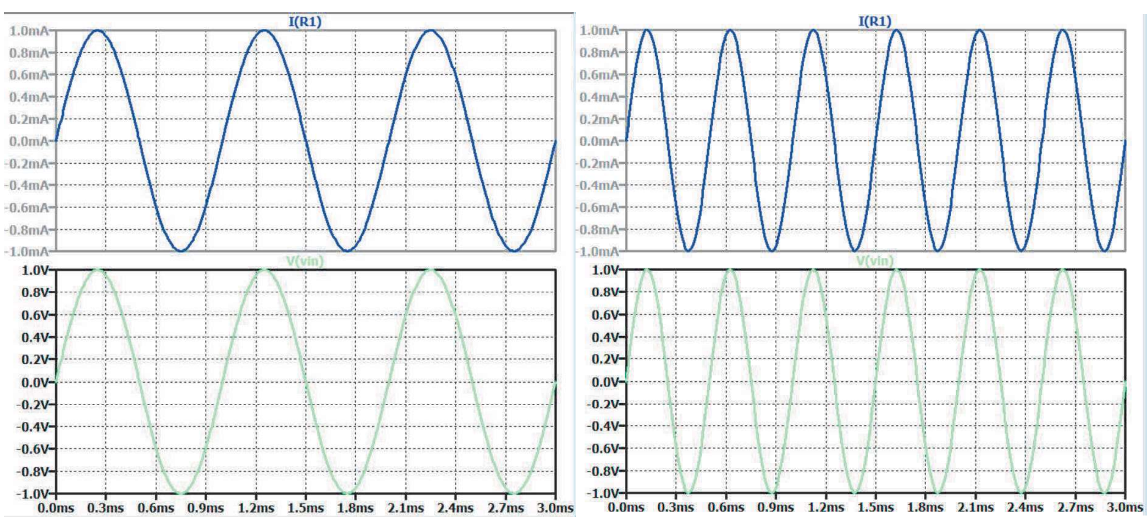


The current through the resistor is given by Ohm's law:

$$i = \frac{V}{R}$$

$$i = \frac{\sin(2\pi f \cdot t)}{R}$$

In the following waveforms we plot the voltage and current across the resistor for R = 1kΩ, and frequencies of 1kHz and 2kHz. Notice that the amplitude of the current is constant over varying frequency.

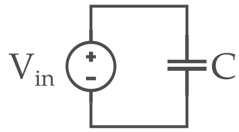


(a) Frequency = 1kHz (b) Frequency = 2kHz

Figure 7.4: V and I on resistor

7.3.2 Capacitors

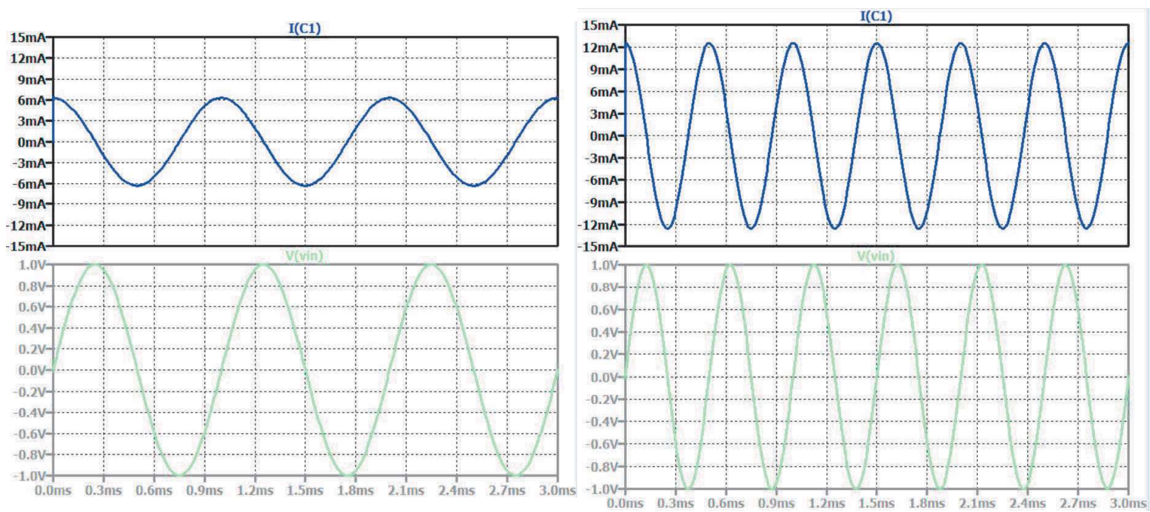
Now let's try a capacitor:



The current is given by the capacitor equation:

$$i = C\frac{dV_{in}}{dt}$$

$$i = C \cdot 2\pi f \cos(2\pi f \cdot t)$$



(a) Frequency = 1kHz, I_{max} = 6.3mA

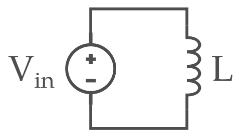
(b) Frequency = 2kHz, I_{max} = 12mA

Figure 7.5: V and I on capacitor

Notice that while the current always has exactly the same frequency as the voltage signal, the amplitude can be different. At low frequencies, the capacitor has very little current flowing through it, as if it were a large resistor. At high frequencies, larger amounts of current flow, as if the resistance is now smaller.

7.3.3 Inductors

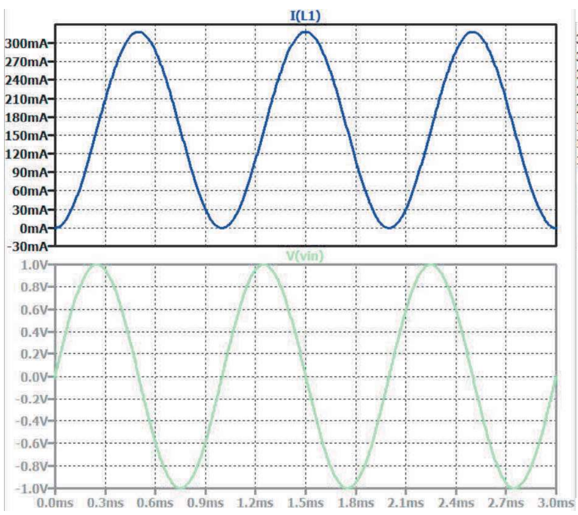
And finally, let's do an inductor:



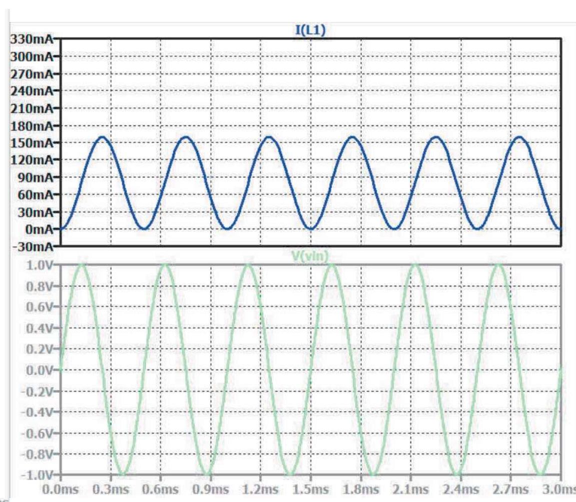
Here the current is

$$i = \frac{1}{L}\int V_{in}dt$$

$$i = -\frac{1}{L \cdot 2\pi f}\cos(2\pi f \cdot t)$$



(a) Frequency = 1kHz, $I_{max} =$ 320mA



(b) Frequency = 2kHz, $I_{max} =$ 160mA

Figure 7.6: V and I on inductor

Just like the capacitor, the frequency of the current is the same as the voltage signal, but the amplitude varies depending on the frequency. An inductor behaves like the dual of a capacitor: presenting a small resistance at low frequencies (higher current for same voltage amplitude), and a large resistance at high frequencies (lower current for same voltage amplitude).

7.3.4 Impedance of R, L, C

Since the frequency of the current is always the same as the frequency of the voltage, we can still define the voltage to current ratio, even for capacitors and inductors. In equations, impedance is represented as "Z", and like resistance it is measured in Ohms. By definition:

$$Z = \frac{V}{I}$$

For a resistor, the impedance is equivalent to the resistance, therefore:

$$Z_R = \frac{V}{I} = R$$

We can also develop equations which describe the impedance of capacitors and inductors. **The phase shift of the signals is taken into account with the symbol *j*** (phase shift being the fact that the fact that the input is a sin() function and the current is cos() - if you are interested, a thorough discussion of how this works is given in the bonus material at the end). By doing this, we can derive some equations describing the impedances of capacitors and inductors.

From our observations, we know that for $V_{in} = 1 \cdot \sin(2\pi f \cdot t)$, the current through the capacitor should be:

$$i_{capacitor} = C \cdot 2\pi f \cos(2\pi f \cdot t)$$
$$Z_{capacitor} = \frac{V_{in}}{i_{capacitor}} = \frac{1}{j2\pi f C}$$

Similarly for the inductor:

$$i_{inductor} = -\frac{1}{L \cdot 2\pi f} \cos(2\pi f \cdot t)$$
$$Z_{inductor} = \frac{V_{in}}{i_{inductor}} = \frac{1}{\frac{1}{L \cdot j2\pi f}} = j2\pi f \cdot L$$

These are general equations which hold true for describing the impedance of capacitors and inductors over all frequencies.

Example: As an exercise, now use these equations to calculate the impedance of a $1\mu F$ capacitor at 1kHz. Using this, we can calculate the current we expect to be going through the capacitor.

$$Z_c = \frac{1}{2\pi \cdot 1kHz \cdot 1\mu F} = 159.15\Omega$$
$$I_c = \frac{1}{159.15} = 6.3mA$$

Notice this matches the value shown on Figure 7.5a.

Now, let us calculate the current flowing through this same capacitor at 100kHz. It increases, as we expect.

$$Z_c = \frac{1}{2\pi \cdot 100kHz \cdot 1\mu F} = 1.59\Omega$$
$$I_c = \frac{1}{159.15} = 630mA$$

Now do repeat the above two exercises for the inductor example - L=1mH, at 1kHz. Check it against Figure 7.6a. Now calculate the current at 100kHz. Does the current increase or decrease with frequency? Is it what you expect? [2]

Questions:

[2] Current = 1.6mA, it has decreased with frequency. This is expected since the impedance of an inductor increases with frequency.

Based on these equations, what can you say about the impedance of a capacitor at DC (Frequency = 0)? What can you say about the impedance of a capacitor at very high frequencies (Frequency = ∞) ? Do they look like short circuits or open circuits?

What can you say about the impedance of an inductor at DC? How about at very high frequencies?

Refer to [3] to check your answers.

7.3.5 Summary

The impedances of resistors, capacitors and inductors can be described by the following equations. Since the term $2\pi f$ appears so often, it is often represented simply as *omega*. You might also often see s, which represents $j2\pi f$.

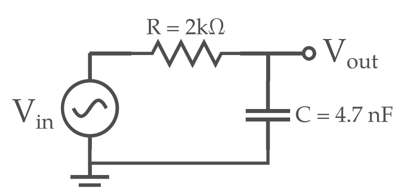
$$
\begin{aligned}
Z_{resistor} &= R \\
Z_{capacitor} &= \frac{1}{j2\pi f C} = \frac{1}{j\omega C} == \frac{1}{sC} \\
Z_{inductor} &= j2\pi f L = j\omega L = sL \\
where \quad & \omega = 2\pi f, \quad s = j2\pi f
\end{aligned}
$$

By using impedance, we can treat capacitors and inductors like resistors when analysing them in the circuit, where the circuit behavior is now frequency dependent.

7.4 Filters - Transfer Functions and Bode Plots

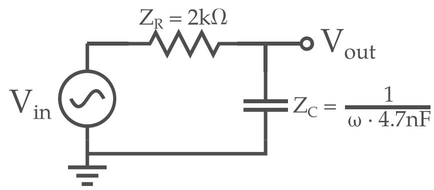
Knowing now that by adding capacitors and inductors, we add frequency dependence to a circuit, we can now explore ways of using them to do more interesting things with our circuits. By using the idea of impedance, we can analyze the circuit using the tools we developed to analyze resistor circuits in the first part of the class. You can use nodal analysis, series/parallel reduction, voltage/current dividers, etc.

Let us use this new method to analyse the RC circuit below, to understand how its behavior depends on frequency. That is, for any sine wave that we put in, we want to find the amplitude of the corresponding output. This is known as the *frequency response*, or sometimes as the *transfer function*.



The impedance of a capacitor is $Z_C = \frac{1}{sC}$. The impedance of a resistor is simply $Z = R$.

[3] A capacitor looks like an open circuit at DC, and a short circuit at very high frequencies. An inductor looks like a short circuit at DC, and an open circuit at very high frequencies.



Now that the circuit is expressed in terms of impedance, the output voltage is just the result of a voltage divider:

$$V_{out} = V_{in} \cdot \frac{Z_C}{Z_R + Z_C}$$

The ratio between the output and input voltages, known as the *gain*, is therefore just

$$\text{Gain} = \frac{V_{out}}{V_{in}} = \frac{Z_C}{Z_R + Z_C}$$

Plugging in the values for this circuit, we can write

$$\text{Gain} = \frac{\frac{1}{s \cdot 4.7\,\text{nF}}}{2\,\text{k}\Omega + \frac{1}{s \cdot 4.7\,\text{nF}}}$$

And multiplying through by $s \cdot 4.7\,\text{nF}$ gives

$$\text{Gain} = \frac{1}{2\,\text{k}\Omega \cdot s \cdot 4.7\,\text{nF} + 1}$$

It's worth making a couple observations at this point. First, the gain will never be more than 1. This makes sense, because the output is the result of a voltage *divider* and must be some fraction of the input. Second, the gain decreases as frequency (s $= j2\pi f$) increases. In other words, as the frequency increases, the capacitor impedance decreases, and the output amplitude becomes less and less.

In other words, this circuit behaves like a *low pass filter*. When a set of signals of varying frequencies but the same amplitude are fed into this circuit, the low frequency signals appear at V_{out} at higher amplitudes, and high frequency signals appear at V_{out} at lower amplitudes (in other words, the amplitude of the signal decreases as frequency increases).

Now that we have a transfer function describing the gain of this circuit, let's plot it on a Bode plot, which is what we use to represent these gain-frequency relationships.

7.4.1 Plotting the transfer function

First, let's use a brute-force approach to plotting, and then we'll work backward to the intuition, and then work out a quick way to plot the frequency response without a computer (and discover why Bode plots are so useful).

Gain is usually expressed in decibels (dB), so we need to convert our gain equation to dB. Remember that $\text{Gain}_{\text{dB}} = 20 \cdot \log_{10}(V)$.

Pull out your favorite plotting tool and plot the gain, using a logarithmic X scale for frequency and a linear Y scale for dB (since dB is already a log scale). Python/NumPy and MATLAB examples are at the end of this document.

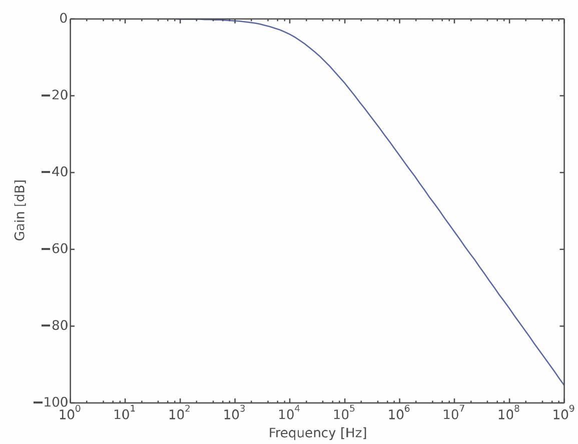


Figure 7.7: Bode plot of simple RC circuit

The gain basically has two straight lines, connected by a smooth curve. The first line is at 0 dB and goes from DC (0 Hz) to about 1 kHz. The second line is a downward slope where the gain steadily drops as frequency increases.

You can see that a good approximation of this gain plot is simply two straight lines connected together. The point at which these two straight lines would intersect is another important feature of the Bode plot. It is a corner, and therefore we call it a **corner frequency**. Bode plots can have multiple corner frequencies - in the case of our example above, there is only one.

When drawing a Bode plot, the first step we usually take is to find this corner frequency.

Finding the corner frequency:
To do this we first write down the gain equation:

$$\text{Gain}_{\text{dB}} = 20 \cdot \log_{10}(|\frac{1}{j2\pi f \cdot 2\,\text{k}\Omega \cdot 4.7\,\text{nF} + 1}|)$$

Since the gain in dB simply looks at the magnitude, we can also drop the j (we are making an approximation, and this is something which enables us to simply the maths for doing this, and is in fact a good approximation except when we are right at the corner) and write:

$$\text{Gain}_{\text{dB}} = 20 \cdot \log_{10}(\frac{1}{2\pi f \cdot 2\,\text{k}\Omega \cdot 4.7\,\text{nF} + 1})$$

To find the corner frequency, we set:

$$2\pi f_c \cdot 2\,\text{k}\Omega \cdot 4.7\,\text{nF} = 1$$

$$f_c = \frac{1}{2\pi \cdot 2\,\text{k}\Omega \cdot 4.7\,\text{nF}} = 16.9\,\text{kHz}$$

Why is this considered the corner frequency? Well, because for frequencies *less* than f_c, we can make the approximation that the $j2\pi f$ term is smaller than 1, resulting in the flat line we draw to the left of the corner frequency. And for the frequencies *more* than f_c, we can make the approximation that the $j2\pi f$ term is larger than 1, resulting in the sloped line that we draw to the right of the corner frequency in this example.

To see this in practice, first assume that $j2\pi f$ is small and therefore the $j2\pi f$ term is negligible. The equation simplifies in this way:

$$\text{Gain}_{\text{dB}} = 20 \cdot \log_{10}(\frac{1}{j2\pi f \cdot 2\,\text{k}\Omega \cdot 4.7\,\text{nF} + 1})$$

$$\text{Gain}_{\text{dB}} = 20 \cdot \log_{10}(\frac{1}{1})$$

$$\text{Gain}_{\text{dB}} = 20 \cdot (\log_{10}(1) - \log_{10}(1))$$

$$\text{Gain}_{\text{dB}} = 0$$

In other words, we can approximate the gain as 0 dB (equivalent to a gain of 1) when $j2\pi f$ is below the corner frequency. Sometimes this is called *unity gain*.

We can find the slope of the line after the corner frequency mathematically - and in fact you will be often asked to do so for non-zero sloeps, as this is an important feature of a Bode plot.

At higher frequencies, the first term $(j2\pi f \cdot 2k\Omega \cdot 4.7nF)$ is large. It will be much larger than 1, and we can treat the $+1$ as negligible and simply drop it, to write the following:

$$\text{Gain}_{\text{dB}} = 20 \cdot \log_{10}(\frac{1}{2\pi f \cdot 2\,\text{k}\Omega \cdot 4.7\,\text{nF}})$$

$$\text{Gain}_{\text{dB}} = 20 \cdot (\log_{10}(1) - \log_{10}(2\pi f \cdot 2\,\text{k}\Omega \cdot 4.7\,\text{nF}))$$

$$\text{Gain}_{\text{dB}} = 0 - 20 \cdot \log_{10}(2\,\text{k}\Omega \cdot 4.7\,\text{nF}) - 20 \cdot \log_{10}(2\pi f)$$

$$\text{Gain}_{\text{dB}} = -20 \cdot \log_{10}(2\pi f) - 20 \cdot \log_{10}(2\,\text{k}\Omega \cdot 4.7\,\text{nF})$$

All the terms in the above equation are constants except for$-20 \cdot \log_{10}(j2\pi f)$, indicating that the slope is decreasing by 20dB for every 10x increase in frequency. Notice that the slope of this line doesn't depend on the component values. You may often hear electrical engineers referring to this as "20dB per decade".

This is a pretty important result - you will find that the slope is almost always some multiple of 20dB per decade.

Another approach to drawing Bode plots
Let's look at the equation for the RC low pass circuit again.

$$\text{Gain}_{\text{dB}} = 20 \cdot \log_{10}(\frac{1}{\omega \cdot 2\,\text{k}\Omega \cdot 4.7\,\text{nF} + 1})$$

We can rewrite this as:

$$Gain_{dB} = 20 \cdot \log_{10}(\frac{1}{\frac{\omega}{\omega_c} + 1})$$
$$where \quad \omega_c = \frac{1}{2\,\text{k}\Omega \cdot 4.7\,\text{nF}} = 2\pi f_c$$
$$therefore \quad f_c = \frac{1}{2\pi \cdot 2\,\text{k}\Omega \cdot 4.7\,\text{nF}} = 16.9kHz$$

This is the corner frequency we found before! This is, in fact, not a coincidence.
Let's consider what happens to the gain when $\omega = \omega_c$.

$$Gain_{dB} = 20 \cdot \log_{10}(\frac{1}{\frac{\omega_c}{\omega_c} + 1})$$

You can see that this is going to be close to unity gain. If $\omega << \omega_c$, then $\frac{\omega}{\omega_c} << 1$ and:

$$Gain_{dB} \approx 20 \cdot \log_{10}(\frac{1}{1}) = 0dB$$

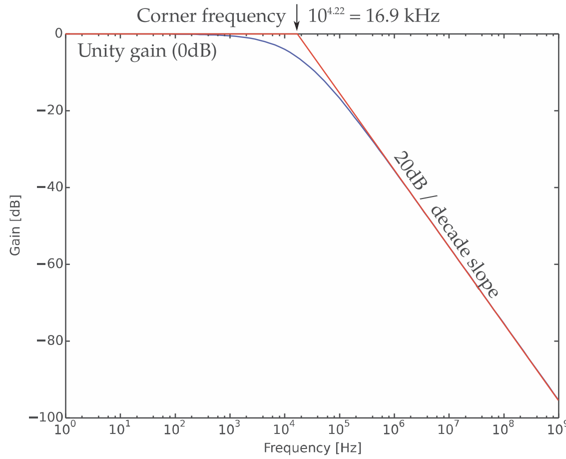
In other words, below the corner frequency, the gain will be unity.

If $\omega >> \omega_c$, then $\frac{\omega}{\omega_c} >> 1$ and:

$$Gain_{dB} \approx 20 \cdot \log_{10}(\frac{1}{\frac{\omega}{\omega_c}}) = 20 \cdot \log_{10} \omega_c - 20 \cdot \log_{10}(\omega)$$

In other words, above the corner frequency, the gain will decrease at a constant slope of -20dB per decade starting from the value of the gain at the corner frequency.

Using these three bits of information: the corner frequency, the gain below the corner frequency, and the gain above the corner frequency, we can draw an approximate Bode plot representation as shown in the figure below in pink. You'll notice this approximation very closely follows the plot which was drawn using computer software, and was actually fairly simple to create.



**An alternative way of seeing the 20dB/dec slope is to plot a few points on the graph for yourself. Choose points which are a decade apart - eg. let's choose points $\omega = 10\omega_c, 100\omega_c, 1000\omega_c$.

$$Gain_{dB}(\omega) = 20 \cdot \log_{10}(\frac{1}{1+\frac{\omega}{\omega_c}})$$

$$Gain_{dB}(10\omega_c) = 20 \cdot \log_{10}(\frac{1}{1+\frac{10\omega_c}{\omega_c}}) = 20 \cdot \log_{10}(\frac{1}{1+10}) \approx 20 \cdot log(0.1) = -20dB$$

$$Gain_{dB}(100\omega_c) = 20 \cdot \log_{10}(\frac{1}{1+\frac{100\omega_c}{\omega_c}}) = 20 \cdot \log_{10}(\frac{1}{1+100}) \approx 20 \cdot log(0.01) = -40dB$$

$$Gain_{dB}(100\omega_c) = 20 \cdot \log_{10}(\frac{1}{1+\frac{1000\omega_c}{\omega_c}}) = 20 \cdot \log_{10}(\frac{1}{1+1000}) \approx 20 \cdot log(0.001) = -60dB$$

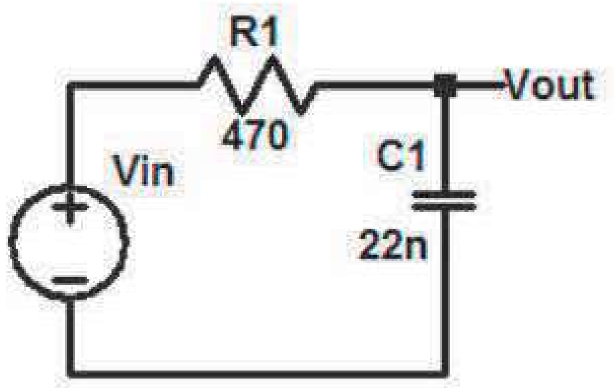
This information is summarised in Table 7.1, and it becomes obvious that after the corner frequency, for every decade increase in frequency, the gain decreases by 20dB. If you were to plot these on a graph, you would end up with a straight line of -20dB/dec slope!

Table 7.1: Plotting a Bode plot using pointss

$\omega(rad/s)$	$Frequency(Hz)$	$Gain_{db}(dB)$
$10\omega_c$	$169kHz$	$-20dB$
$100\omega_c$	$1.69MHz$	$-40dB$
$1000\omega_c$	$16.9MHz$	$-60dB$

An example

Find the transfer function of the following circuit, and its corner frequency. Plot the transfer function of this circuit on a Bode plot, indicating its corner frequency, and the value of any slopes.



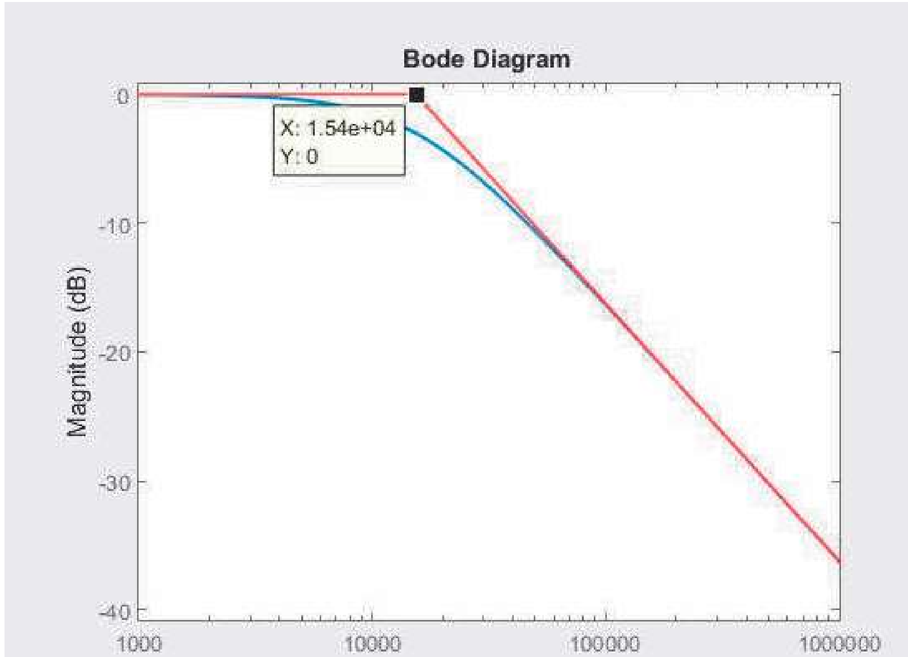
$$Z_R = R = 470\Omega$$
$$Z_C = \frac{1}{\omega C} = \frac{1}{\omega 22nF}$$
$$Gain = \frac{V_{out}}{V_{in}} = \frac{Z_C}{Z_R + Z_C} = \frac{\frac{1}{\omega \cdot 22nF}}{470\Omega + \frac{1}{\omega \cdot 22nF}} = \frac{1}{1 + \omega \cdot 22nF \cdot 470\Omega}$$
$$\text{Rewrite as: } Gain = \frac{1}{1 + \frac{\omega}{\omega_c}} \qquad where \quad \omega_c = \frac{1}{22nF \cdot 470\Omega}$$
$$\text{Therefore: } f_c = \frac{1}{2\pi \cdot 22nF \cdot 470\Omega} = 15.4kHz$$

Now that we know the corner frequency, we need to find the slopes of the two lines that meet at the corner frequency.

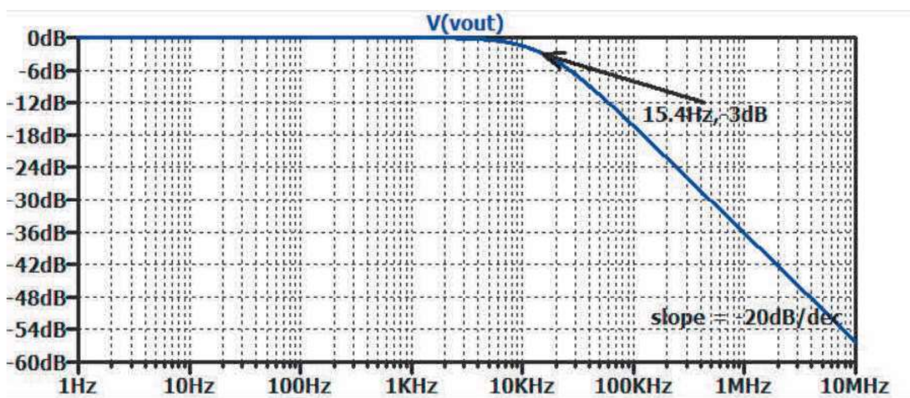
$$Gain_{dB} = 20 \cdot \log_{10}(\frac{1}{1 + \frac{\omega}{\omega_c}})$$
When $\omega << \omega_c$, then $\frac{\omega}{\omega_c} << 1$ and:
$$Gain_{dB} \approx 20 \cdot \log_{10}(\frac{1}{1}) = 0dB$$
So we draw a straight line of 0dB gain up to the corner frequency.

When $\omega >> \omega_c$, then $\frac{\omega}{\omega_c} >> 1$ and:
$$Gain_{dB} \approx 20 \cdot \log_{10}(\frac{1}{\frac{\omega}{\omega_c}}) = 20 \cdot \log_{10} \omega_c - 20 \cdot \log_{10}(\omega)$$
So we draw a straight line of -20dB/dec above the corner frequency.
And these two lines will intersect at the corner frequency.

The plot can be drawn simply using asymptotes as indicated in red on the following figure:



A simulated plot would look like:



7.4.2 More RC circuits

Let's consider another RC filter, this time configured a little differently. We want to find its frequency response/transfer function, and plot this on a Bode plot.

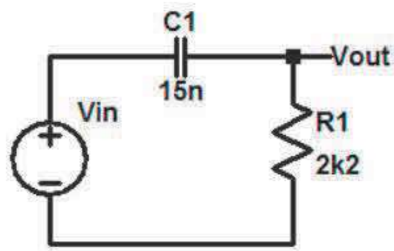


Figure 7.8: Another RC circuit

First, consider this circuit qualitatively. At low frequencies, the capacitor presents a very large resistance. Hence we expect $\frac{V_{out}}{V_{in}}$ to be small at low frequencies. Conversely, at high frequencies the capacitor presents a very small resistance. Therefore we expect $\frac{V_{out}}{V_{in}}$ to be high (approaching unity as the capacitor approaches becoming a short circuit). Now let's look at this quantitatively and see if the results match what we expect. Start by finding the impedance of the capacitor and the resistor.

$$Z_R = R = 2.2k\Omega$$
$$Z_C = \frac{1}{\omega C} = \frac{1}{\omega 15nF}$$
$$\frac{V_{out}}{V_{in}} = \frac{Z_R}{Z_R + Z_C} = \frac{2.2k\Omega}{2.2k\Omega + \frac{1}{\omega \cdot 15nF}}$$
$$Gain = \frac{V_{out}}{V_{in}} = \frac{\omega \cdot 15nF \cdot 2.2k\Omega}{\omega \cdot 15nF \cdot 2.2k\Omega + 1}$$

By simply looking at the above equation, can you see that the gain approaches zero at low frequencies and unity at high frequencies?

Let's rewrite the equation so that it's easier to see this.

$$Gain = \frac{V_{out}}{V_{in}} = \frac{\frac{\omega}{\omega_c}}{\frac{\omega}{\omega_c} + 1} \qquad \text{where} \qquad \omega_c = \frac{1}{RC} = \frac{1}{2.2k\Omega \cdot 15nF}$$

$$\text{Therefore the corner frequency } f_c = \frac{1}{2\pi \cdot 2.2k\Omega \cdot 15nF} = 4.8kHz$$

When $\omega >> \omega_c$, then $\frac{\omega}{\omega_c} >> 1$ and:

$$Gain_{dB} \approx 20 \cdot \log_{10}(\frac{\frac{\omega}{\omega_c}}{\frac{\omega}{\omega_c}}) = 20 \cdot \log_{10}(1) = 0dB$$

Therefore, this RC circuit exhibits unity gain above the corner frequency (as we expected!)

If $\omega << \omega_c$, then $\frac{\omega}{\omega_c} << 1$ and:

$$Gain_{dB} \approx 20 \cdot \log_{10}(\frac{\omega}{\omega_c}) = 20 \cdot \log_{10}(\omega) - 20 \cdot \log_{10}(\omega_c)$$

Below the corner frequency, the gain increases at 20dB/dec up to the value of the gain at the corner frequency. Using this information, we can draw two asymptotes crossing at the corner frequency to represent the Bode plot of this circuit. This is shown in Figure 7.9.

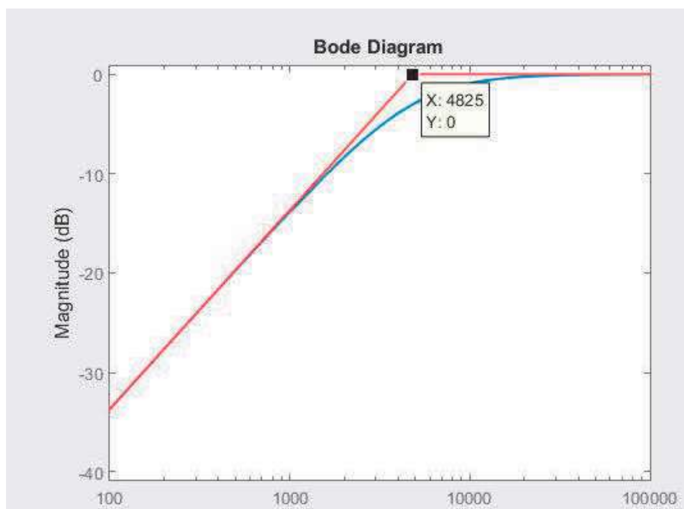


Figure 7.9: Simple Bode plot of the high pass filter

By swapping the capacitor and the resistor around, we have created a high pass filter (low gain at low frequencies, and unity gain at higher frequencies).

An accurate Bode plot (drawn in a simulator) is shown in Figure 7.10. If we were to superimpose the two on top of each other, you would find a significant deviation only at the corner frequency (which has a gain of -3dB rather than 0dB).

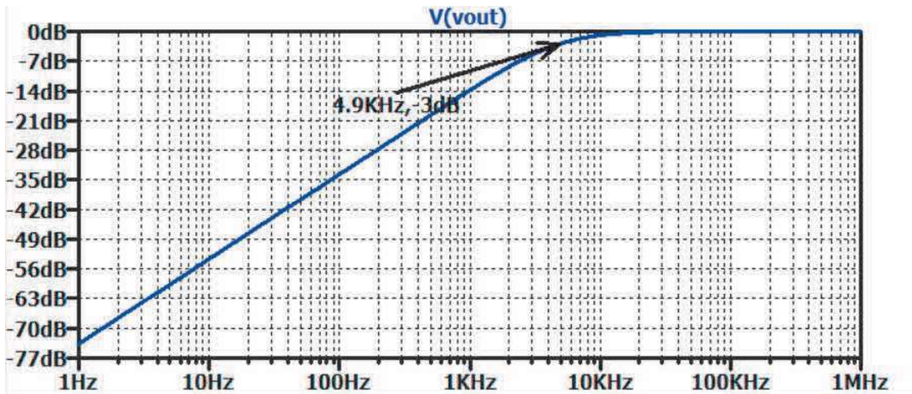


Figure 7.10: Bode plot of the high pass filter

Finally, let's look at a more complex RC circuit. This circuit has two capacitors.

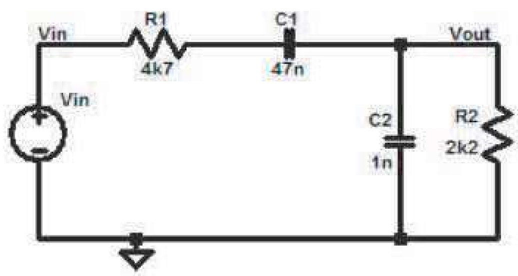


Figure 7.11: A slightly more complex RC circuit

$$Z_R1 = R1 = 4.7k\Omega$$

$$Z_R2 = R2 = 2.2k\Omega$$

$$Z_C1 = \frac{1}{\omega C1} = \frac{1}{\omega \cdot 47nF}$$

$$Z_C2 = \frac{1}{\omega C2} = \frac{1}{\omega \cdot 1nF}$$

Let's call the series combination of R2 and C2 a lump impedance Z2, and the parallel combination of R1 and C1 a lump impedance Z1, as shown in Figure 7.12.

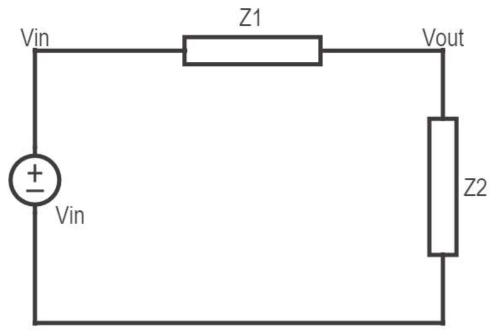


Figure 7.12: Abstracting the circuit to obtain the transfer function

Then the circuit is simply a voltage divider:

$$Gain = \frac{V_{out}}{V_{in}} = \frac{Z2}{Z2 + Z1}$$

Writing down the actual impedance for Z1 and Z2:

$$Z1 = Z_{R1} + Z_{C1} = R1 + \frac{1}{\omega \cdot C1} = \frac{1 + \omega \cdot R1 \cdot C1}{\omega \cdot C1}$$

$$Z2 = Z_{R2} \parallel Z_{C2} = \frac{1}{\frac{1}{R2} + \omega \cdot C2} = \frac{R2}{1 + \omega \cdot R2 \cdot C2}$$

Substituting these back into the gain equation we get:

$$Gain = \frac{\frac{R2}{1 + \omega \cdot R2 \cdot C2}}{\frac{1 + \omega \cdot R1 \cdot C1}{\omega \cdot C1} + \frac{R2}{1 + \omega \cdot R2 \cdot C2}}$$

While this looks bad, it will look much better after a little algebra. Multiply numerator and denominator by $(\omega C1) \cdot (1 + \omega \cdot R2 \cdot C2)$ to to get rid of the fractions on the bottom:

$$= \frac{\omega \cdot R2 \cdot C1}{(1 + \omega \cdot R1 \cdot C1) \cdot (1 + \omega \cdot R2 \cdot C2) + \omega \cdot R2 \cdot C1}$$

While this is a little more complex than the previous gain formulas, we can still find the lines that make up the gain plot, by systematically looking at the equation at in different frequency bands separated by the corner frequencies we identify: $\omega_{c1} = \frac{1}{R1 \cdot C1} = 4.53krad/s = 722Hz$, and $\omega_{c2} = \frac{1}{R2 \cdot C2} = 454.5krad/s = 72.3kHz$

- At low frequencies: When F (ω) is very small, the denominator will be about 1 ($1 >> \omega R2 \cdot C2$), so the gain will be $2\pi F \cdot R2 \cdot C1 = F \cdot 650\mu$s. This means at low frequencies, we will draw a line increasing at 20dB/dec, up to the first corner frequency ω_{c1}

- Next, we look at what happens when the frequency is above ω_{c1} but below ω_{c2}. Then the equation would be approximated by the following, since $\omega R1 \cdot C1 >> 1$, $\omega R2 \cdot C2 << 1$:

$$\approx \frac{\omega \cdot R2 \cdot C1}{(\omega \cdot R1 \cdot C1) \cdot (1) + \omega \cdot R2 \cdot C1}$$
$$= \frac{\omega R2}{\omega R1 + \omega R2} = \frac{R2}{R1 + R2} = -9.9dB$$

  It is just a resistive divider! Hence, in this range, the circuit is simply a flat line (slope = 0) at a magnitude of -9.9dB.

- Finally, we consider what happens when the frequency is above $\omega R2$. Then the equation would be approximated by the following, since $\omega R1 \cdot C1 >> 1$, $\omega R2 \cdot C2 >> 1$:

$$\approx \frac{\omega \cdot R2 \cdot C1}{(\omega \cdot R1 \cdot C1) \cdot (\omega \cdot R2 \cdot C2) + \omega \cdot R2 \cdot C1}$$
$$= \frac{1}{\omega \cdot R1 \cdot C2 + 1} \quad \text{Then, since } \omega \cdot R1 \cdot C2 >> 1$$
$$\approx \frac{1}{\omega \cdot R1 \cdot C2}$$

  This means at high frequencies (above ω_{c2}) we will draw a line decreasing at -20dB/dec.

The final Bode plot would look like the following. It is essentially a bandpass filter.

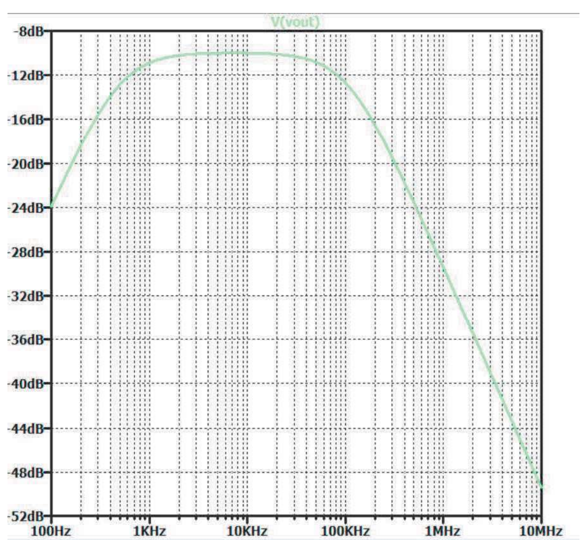


Figure 7.13: Bode plot of more complex RC circuit

7.4.3 Summary

You have now seen how to create and analyze a low pass filter, high pass filter, and bandpass filter.

To summarize, when you are given an RLC circuit and asked to find the frequency response:

1. Find the impedance for each element in the circuit.
2. Solve for the output in terms of the input to get the gain.
3. Convert to dB and plot.
4. The resulting plot can be approximated by a set of straight lines where it is easy to estimate one point of the line, and its slope. If the gain is changing with F the slope with be 20dB/decade, and if it is changing by 1/F it is -20dB/decade. If the gain is changing by F^2, then it will be 40dB/decade (since in log scale, squaring something just multiplies it by 2).
5. This straight line approximation will provide the all the information you need, including the corner frequencies. The position of the lines and the corner frequencies does not change when you use the "correct" formulas that incorporate phase information.
6. Of course if you want an exact plot you should use a computer.

7.4.4 Practice Examples: RC circuits and Bode plots

For each of these circuits, derive the transfer function and sketch the Bode plot of the frequency response. Use what you've learnt about capacitor behaviour at high and low frequencies to check that your answers make sense.

Problem 7.1

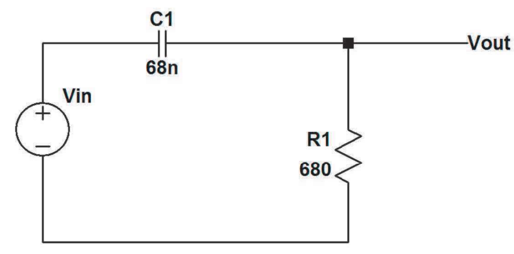


Figure 7.14: RC circuit example 1

Problem 7.2

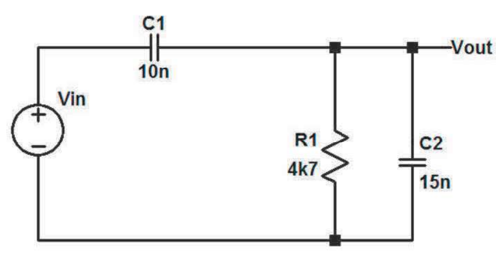


Figure 7.15: RC circuit example 2

7.4.5 Solutions to practice examples

Solution 7.1:

$$Z_R = R = 680\Omega$$
$$Z_C = \frac{1}{\omega C} = \frac{1}{\omega 68nF}$$
$$\frac{V_{out}}{V_{in}} = \frac{Z_R}{Z_R + Z_C} = \frac{680\Omega}{680\Omega + \frac{1}{\omega \cdot 68nF}}$$
$$Gain = \frac{V_{out}}{V_{in}} = \frac{\omega \cdot 68nF \cdot 680\Omega}{\omega \cdot 68nF \cdot 680\Omega + 1} = \frac{\frac{\omega}{\omega_c}}{\frac{\omega}{\omega_c} + 1}$$
$$\text{Therefore } \omega_c = \frac{1}{68nF \cdot 680\Omega}$$

Therefore the corner frequency $f_c = \frac{\omega_c}{2\pi} = \frac{1}{2\pi \cdot 68nF \cdot 680\Omega} = 3.44kHz$. From the gain equation we can see that the gain approaches zero at low frequencies, and approaches unity at high frequencies. Using this information, we can construct a simple Bode plot using straight lines as shown.

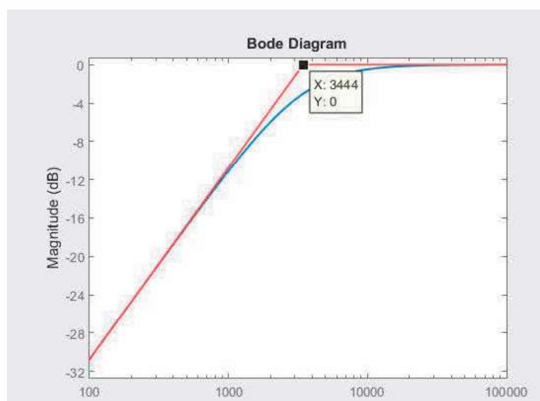


Figure 7.16: RC circuit example 1 - Simplified hand drawn Bode plot

An accurate Bode plot is shown in Figure 7.17.

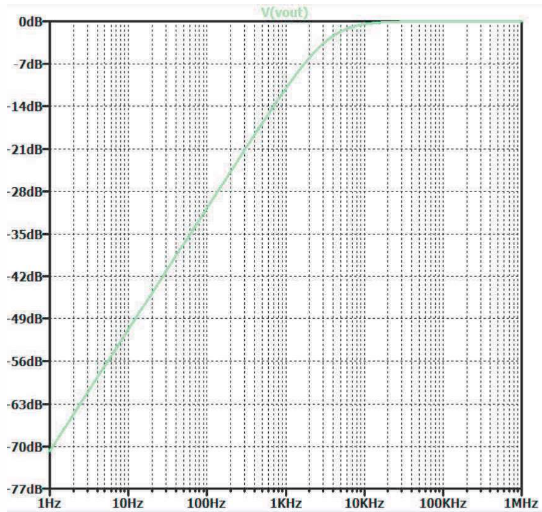


Figure 7.17: RC circuit example 1 - Bode plot

Solution 7.2:
If we just think about the circuit qualitatively, it be thought of as a voltage divider comprising two elements - Z_{C1} and $Z_R \parallel Z_{C2}$. In other words:

$$\frac{V_{out}}{V_{in}} = \frac{Z_R \parallel Z_{C2}}{Z_{C1} + Z_R \parallel Z_{C2}}$$

The parallel combination of R1 and C2 is dominated by R1 at low frequencies, since C2 appears to be an open circuit. Similarly, C1 appears as an open circuit (or a very big impedance) and therefore the fraction of Vin that appears at Vout will be very small at low frequencies.

We can then derive the transfer function and Bode plots, and check them by seeing if they match our intuition of how the circuit should behave.

$$Z_R = R = 4.7k\Omega$$
$$Z_{C1} = \frac{1}{\omega C1} = \frac{1}{\omega 10nF}$$
$$Z_{C2} = \frac{1}{\omega C2} = \frac{1}{\omega 15nF}$$
$$Z_R \parallel Z_{C2} = \frac{Z_R \cdot Z_{C2}}{Z_R + Z_{C2}} = \frac{R \cdot \frac{1}{\omega C1}}{R + \frac{1}{\omega C1}} = \frac{R}{1 + \omega \cdot R \cdot C2}$$
$$Gain = \frac{V_{out}}{V_{in}} = \frac{\frac{R}{1+\omega \cdot R \cdot C2}}{\frac{1}{\omega C1} + \frac{R}{1+\omega \cdot R \cdot C2}} = \frac{1}{\frac{1}{\omega C1} \cdot \frac{1+\omega \cdot R \cdot C2}{R}} = \frac{1}{1 + \frac{1+\omega \cdot R \cdot C2}{\omega \cdot R \cdot C1}} =$$

$$\frac{\omega \cdot R \cdot C1}{1 + \omega \cdot R \cdot C1 + \omega \cdot R \cdot C2} = \frac{\omega \cdot R \cdot C1}{1 + \omega \cdot R \cdot (C1 + C2)}$$

The numerator shows that the gain approaches zero at low frequencies. At high frequencies, the 1 becomes insignificant compared to other terms, and the equation simplifies to a capacitive divider, as we expected, ie. $Gain = \frac{C1}{C1 + C2}$.

The denominator is in a form we are familiar with, indicating that there is a corner frequency somewhere, where $\omega_c = \frac{1}{\omega \cdot R \cdot (C1 + C2)}$.

To draw the Bode plot, we need to calculate the corner frequency and the gain at high frequencies.

$$f_c = \frac{\omega_c}{2\pi} = \frac{1}{2\pi \cdot R \cdot (C1 + C2} = 1.35kHz$$

$$Gain_{dB} \text{ at high frequencies} = 20 \cdot \log \frac{C1}{C1 + C2} = 20 \cdot log 0.4 = -7.96dB$$

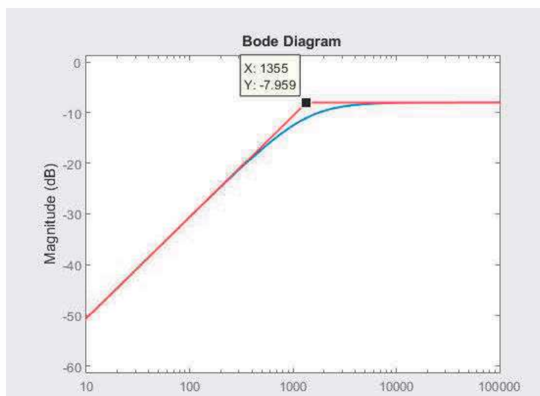


Figure 7.18: RC circuit example 2 - Simplified hand drawn Bode plot

An accurate Bode plot is shown in Figure 7.19.

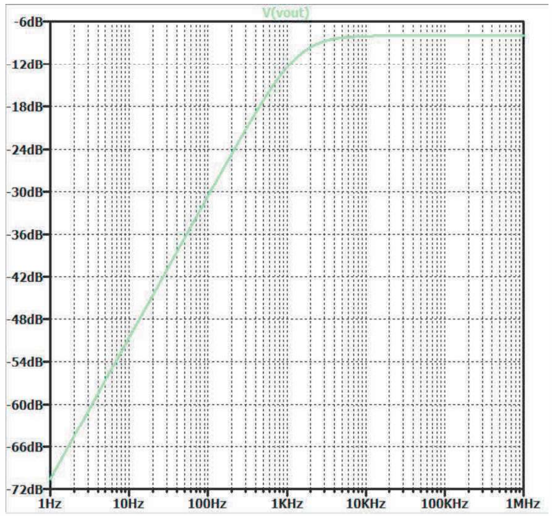
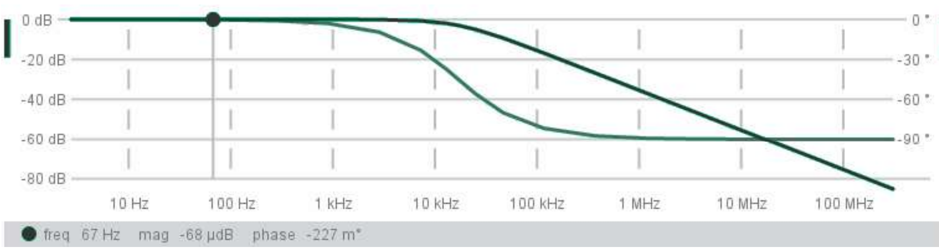


Figure 7.19: RC circuit example 2 - Accurate Bode plot

7.5 Using EveryCircuit

EveryCircuit can do a frequency domain simulation, which is an excellent way to check your answers or gain intuition about how the circuits work.

Build your circuit, and view the voltages at both the input and output nodes (click on the node, then click the "eye" in the lower-left corner). Then click the yellow "Run AC" button, labeled with an 'f'. The bright green line is the gain; the pale green line is the phase. You can zoom and pan the plot if it doesn't show the frequency range you're interested in.



7.5.1 BONUS MATERIAL - Impedance done precisely

In reality capacitors and inductors also introduce a "phase shift" to the signal. If you look at Figures 7.5 and 7.6, you'll notice the current is sinusoidal, but achieves its peak value at a different time from the voltage. This is due to the fact that when you take the integral or a derivative of a sine wave, you end up with a cosine wave, which is a phase shifted version of the sine wave.

In this section we will develop a precise way of calculating impedance which takes into account both the change in amplitude over frequency, and the phase shift.

To get rid of the sine wave to cosine wave transformation, lets consider a different input waveform driving the device: an exponential.

$$V_{in} = V_o e^{st} \qquad \text{where } s = -1/\tau$$
$$i_R = \frac{Vo}{R}\, e^{st}$$
$$i_C = C\frac{d}{dt}V_{in} = sC \cdot V_o e^{st}$$
$$i_L = \frac{1}{L}\int V_{in} dt = \frac{1}{sL} \cdot V_o e^{st}$$

The impedances can then be written as:

$$Z_R = \frac{V_{in}}{i_R} = R$$
$$Z_C = \frac{V_{in}}{i_C} = \frac{1}{sC}$$
$$Z_L = \frac{V_{in}}{i_L} = sL$$

From this it can be observed that if the input voltage is an exponential, solving for the current through capacitors and inductors become very simple, and we have no worries about a phase shift. But we don't really want to drive the components with an exponential, we want to drive it with a sine wave, and at first sine functions and exponential functions seem completely different from each other. But are they?

The definition of an exponential function is that the derivative of the function is equal to the function times a constant:

$$\frac{d}{dt}e^{st} = s \cdot e^{st}$$

So taking the derivative again gives:

$$\frac{d^2}{dt^2}e^{st} = s^2 \cdot e^{st}$$

Now let's look at what happens when we do this with a sine wave:

$$\frac{d}{dt}\sin(\omega t) = \omega \cdot \cos(\omega t)$$

Taking the derivative again gives:

$$\frac{d^2}{dt^2}\sin(\omega t) = -\omega^2 \cdot \sin(\omega t)$$

Notice that if we look at the second derivative lines, to make the sin() and exp() don't look that different. Both functions are unchanged after the second derivative, and both are multiplied by a constant squared. But there is this nasty negative sign in one. To make them more similar, I need to make $s^2 = -\omega^2$. Of course that is impossible if we are dealing with real numbers, but it is easy to do if I can use imaginary numbers. Imaginary numbers are numbers that when squared are negative and are typically written as $x \cdot j$ where $j = \sqrt{-1}$.[4] So the two functions don't look that different if I make $s = j \cdot \omega$?. But what does an exponential with an imaginary time constant mean? To help work some of this out, let's define two functions. The first function, g, is going to an exponential, which gives the expected result when you take its derivative:

$$g(t) = e^{st}$$
$$\frac{d}{dt}g(t) = s \cdot e^{st} = s \cdot g(t)$$

So far no surprises. Now let me define another function h. This function returns a complex number (it has a real part and an imaginary part), and is the sum of a cosine wave and an imaginary sine wave. Taking the derivative of this function is also easy:

$$h(t) = \cos(\omega t) + j \cdot \sin(\omega t)$$
$$\frac{d}{dt}h(t) = \omega \cdot [-sin(\omega t) + j \cdot cos(\omega t)]$$

Now here comes the surprising part. If you look at the derivative of h(t), it turns out to be a constant time h(t). The cos term is multiplied by $j\omega$, and the sin term, was also multiplied by $j\omega$ making it now real and negative. In other words:

$$\frac{d}{dt}h(t) = i\omega \cdot h(t)$$

Notice that this equation is exactly the same equation as the equation for g, if $s = j \cdot \omega$. Said differently, we just figured out what a complex exponential represents:

$$e^{j\omega t} = \cos(\omega t) + j \cdot \sin(\omega t)$$

This is a very famous result in mathematics and is known as Euler's equation! Since cos(-x) = cos (x), and sin(-x) = -sin(x), we have:

$$e^{i\omega t} = cos(\omega t) + j \cdot sin(\omega t); \qquad e^{-i\omega t} = cos(\omega t) - j \cdot sin(\omega t)$$

$$\sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}; \qquad \cos \omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

Thus sinusoidal inputs are really just the sum of two exponential function (with complex time constants), and since the system is linear, we can look at the response to each exponential individually. This is great, since we know the derivative of an exp is always an exponential, and we don't have to worry about sin and cosine waves. So the precise definition of impedance is simply:

[4] Electrical engineers use i to represent current so they typically use j to represent $\sqrt{-1}$, while the rest of the world use $i = \sqrt{-1}$

$$Z_R = \frac{V_{in}}{i_R} = R$$

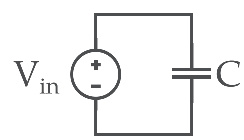
$$Z_C = \frac{V_{in}}{i_C} = \frac{1}{sC}$$

$$Z_L = \frac{V_{in}}{i_L} = sL$$

$$\text{where } s = i\omega = i2\pi f$$

Now while it might have seemed like we went through a lot of math mumbo-jumbo, it might not be obvious what any of this has to do with phase shifts. It turns out that you can plot a complex number on a plane by making the real number the X coordinate and the imaginary number the Y coordinate. When you do this, the angle formed between the point you plot, the origin, and the positive real axis is the phase shift of the system.

Recall the following:



For $V_{in} = sin(2\pi\omega t)$ the current is given by the capacitor equation:

$$i = C\frac{dV_{in}}{dt}$$

$$i = C \cdot \omega \cos(\omega \cdot t)$$

We can see from these equations that the capacitor current will lag the capacitor voltage by 90 degrees ie. the capacitor introduced a -90 degree phase shift.

If we take the precise representation of capacitor impedance:

$$Z_C = \frac{V_{in}}{i_C} = \frac{1}{sC} = \frac{1}{j\omega C}$$

and plot this onto a complex plane, it will be a point on the negative imaginary axis, which exactly represents the -90 degree phase shift.

7.6 Bode plots using Python

```
# Make a bode plot of the frequency response for an RC circuit example
# This ignores phase, so results near the corner frequency are only approximate
# Stanford ENGR 40M

import numpy as np
```

```
import matplotlib.pyplot as plt

f = np.logspace(0, 9, 100) # Logarithmic spacing from 1 to 10^9
w = f * 2 * np.pi # Convert Hz to radians / second

R = 2e3 # 2k ohms
C = 4.7e-9 # 4.7 nanofarads

zR = R # Resistor impedance is just the resistance; doesn't depend on f
zC = 1 / (w * C)

gain = zC / (zR + zC)
db = 20 * np.log10(gain) # Convert voltage gain to dB

plt.semilogx(f, db)
plt.xlabel('Frequency [Hz]')
plt.ylabel('Gain [dB]')

# Optional, plot the corner frequency
corner = 1/(2*np.pi*R*C)
plt.plot([corner, corner], [-100, 0])

plt.show()
```

7.7 Bode plots using MATLAB

```
% Make a bode plot of the frequency response for an RC circuit example
% This ignores phase, so results near the corner frequency are only approximate
% Stanford ENGR 40M

f = logspace(0, 9, 100); % Logarithmic spacing from 1 to 10^9
w = f * 2 * pi; % Convert Hz to radians / second

R = 2e3; % 2k ohms
C = 4.7e-9; % 4.7 nanofarads

zR = R; % Resistor impedance is just the resistance; doesn't depend on f
zC = 1 ./ (w * C);

gain = zC ./ (zR + zC);
db = 20 * log10(gain); % Convert voltage gain to dB

semilogx(f, db);
xlabel('Frequency [Hz]');
ylabel('Gain [dB]');
```