Filters and Bode magnitude plots (*corrected version*)

ENGR 40M lecture notes — August 4, 2017
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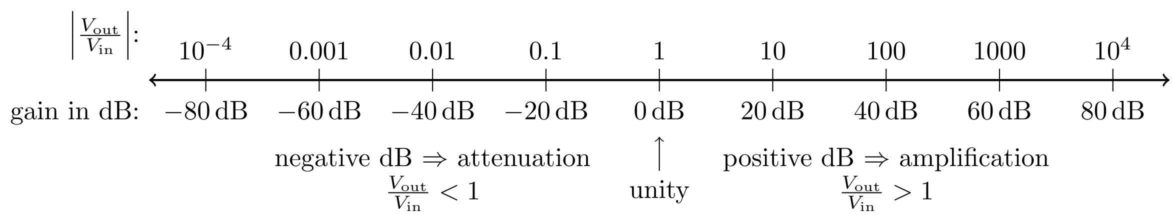
The decibel

Recall that the gain of a circuit is the ratio $\frac{V_{\text{out}}}{V_{\text{in}}}$. We often express (the magnitude of) gains on a logarithmic scale, using a unit called *decibels* (dB). The gain in decibels is defined as

$$\text{gain in dB} = 20\log_{10}\left|\frac{V_{\text{out}}}{V_{\text{in}}}\right|.$$

Expressing gains in decibels allows us to see a much bigger range with reasonable numbers. For example, it's hard to distinguish between 0.0001 and 0.01 on a linear scale, but in decibels they differ by 40 dB.



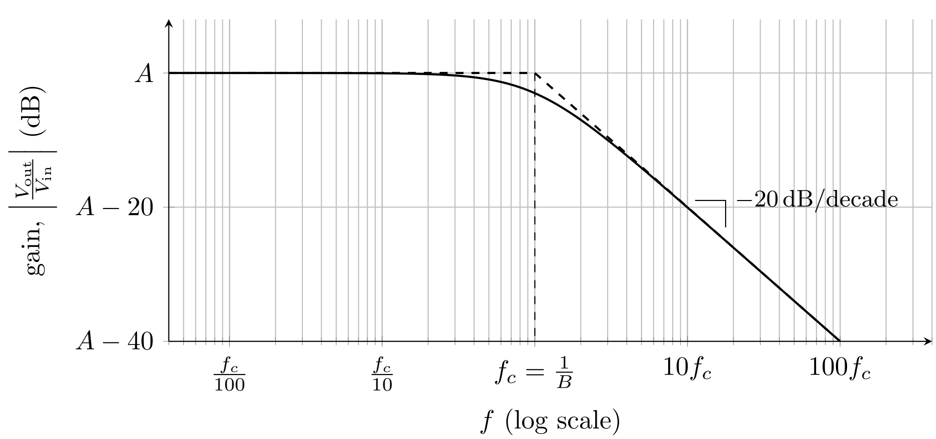
The Bode magnitude plot

The Bode plot is named after Hendrik Wade Bode, an American engineer who proposed it in 1938 as a way to simplify the analysis of systems in the frequency domain. It comprises two plots: one for magnitude and one for phase. In ENGR 40M, we'll only study the magnitude plot.

The Bode magnitude plot is a graph of the *absolute value of the gain* of a circuit, as a function of *frequency*. The gain is plotted in decibels, while frequency is shown on a logarithmic scale. It is therefore a *log–log plot*.

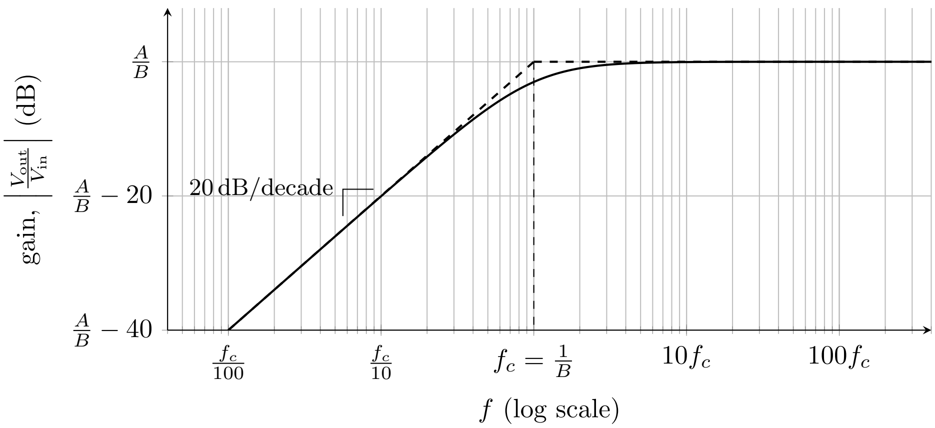
Many systems have a form that makes them very easy to plot on a Bode plot. For example, it's very common for a circuit to have a gain of the form of a *first-order low-pass filter*,

$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{A}{1 + jBf}. \tag{1}$$



Another form commonly encountered is the *first-order high-pass filter*,

$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{Af}{1 + jBf}. \tag{2}$$



Asymptotes

Hendrik Wade Bode's insight was that a log–log plot allows for an *straight-line asymptotic approximation* that is easy to draw and understand. You can find the asymptotes by taking, respectively, $f \to 0$ and $f \to \infty$, and then arguing that $1 \gg Bf$ (in one case) or $1 \ll Bf$ (in the other). When we say that $1 \gg Bf$, we mean that 1 is *much* greater than Bf, which implies that $1 + jBf \approx 1$; and the vice versa for $1 \ll Bf$.

The power of the Bode plot, however, is that you don't need to *derive* these asymptotes. A handful of salient features suffices to make the plot.

Important features

In a Bode plot of a first-order circuit, like the two forms above, there are three salient features that characterize the circuit. Generally, to draw a Bode plot, it suffices to find these three features.

- **Horizontal asymptote.** We find the horizontal asymptote as follows:
  - In a low-pass filter, we take $f \to 0$ to get $\frac{V_{\text{out}}}{V_{\text{in}}} \approx A$. This represents the DC gain.
  - In a high-pass filter, we take $f \to \infty$ to get $\frac{V_{\text{out}}}{V_{\text{in}}} \approx \frac{A}{B}$. This represents the high-frequency gain.
- **Corner frequency (f_c).** Arguably the most defining characteristic, the corner frequency f_c is where the two asymptotes intersect. It can be shown that this is equal to the frequency f at which the two additive terms in the denominator are equal (in magnitude). If you write your gain in the form (1) or (2), this frequency will be $f = \frac{1}{B}$.

  The corner frequency indicates where a filter stops "allowing" frequencies to pass through, *i.e.*, where the gain starts to fall away from the maximum gain (horizontal asymptote).
- **Slope of sloped asymptote.** A "decade" is a multiplying of frequency by 10.[1] Note that each decade is the *same distance* on the log-scaled frequency axis.
  - In a first-order low-pass filter, this is always $-20\,\text{dB/decade}$.
    That is, each time you multiply frequency by 10, you divide gain by 10.
  - In a first-order high-pass filter, this is always $20\,\text{dB/decade}$.
    That is, each time you multiply frequency by 10, you multiply gain by 10.

  Note that, because of how log–log plots work, this slope doesn't depend on A or B.

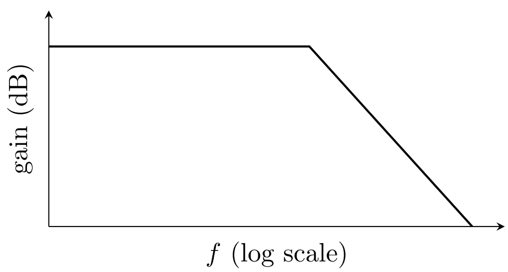
[1] A *doubling* of frequency is called an "octave", named after the musical interval between two notes so related. Then $20\,\text{dB/decade} \approx 6\,\text{dB/octave}$.

Filters

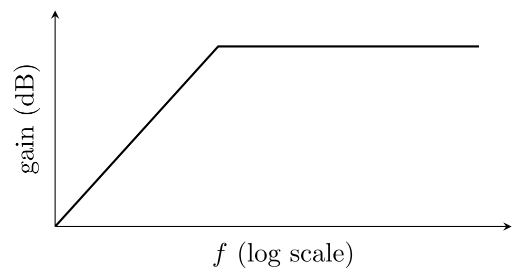
The *low-pass filter* described above in (1) is so named because it allows low frequencies to "pass through" with a gain of A, while attenuating high frequencies. Similarly, the *high-pass filter* in (2) allows high frequencies to pass through with a gain of $\frac{A}{B}$, while attenuating low frequencies.

There are two other types of filters: *band-pass*, which allows frequencies within a (finite) band and attenuates all frequencies outside that band, and *band-stop*, which attenuates frequencies within a (finite) band and allows all others.

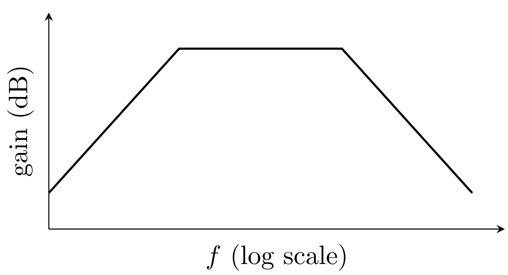
Low-pass filter



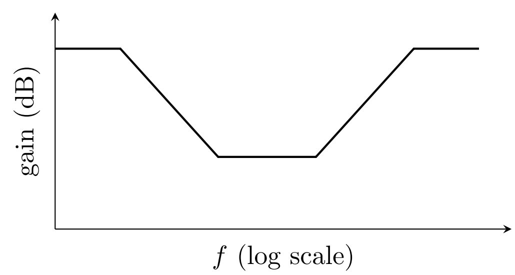
High-pass filter



Band-pass filter



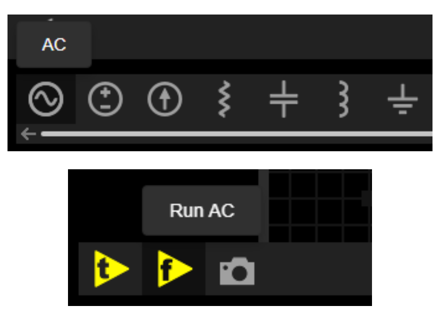
Band-stop filter



Exercise 1. What do you think the *ideal* versions of each of the above types of filters would be? (*Note:* Ideal filters aren't practically realizable.)

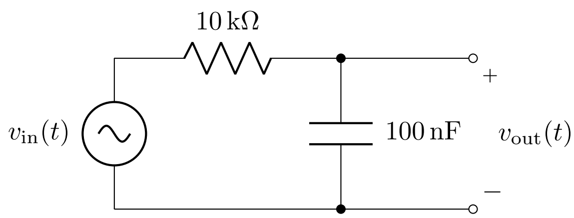
EveryCircuit

If you include an AC source in a circuit on EveryCircuit, you can get EveryCircuit to produce a Bode plot for the circuit by clicking the "Run AC" button. You'll need to probe the output, just as you do for the time-based simulation.



Examples

Example 1. We return to the circuit in Example 1 from last lecture.



Recall that the gain of this circuit was

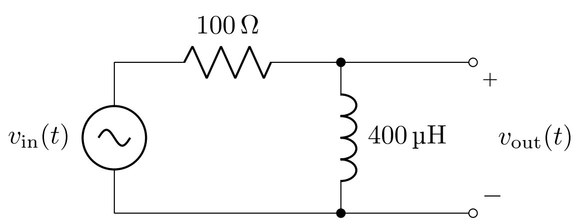
$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{1}{1 + j2\pi fRC} = \frac{1}{1 + jf \cdot 2\pi \times 10^{-3}}.$$

(a) What are the corner frequency f_c and DC gain of this circuit?

(b) Draw a Bode plot for the gain $\frac{V_{\text{out}}}{V_{\text{in}}}$ of this circuit.

(c) What type of filter is this?

Example 2. We return to the circuit in Example 2 from last lecture.



Recall that the gain of this circuit was

$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{j2\pi fL}{R + j2\pi fL} = \frac{jf \cdot 8\pi \times 10^{-4}}{100 + jf \cdot 8\pi \times 10^{-4}}.$$

(a) What are the corner frequency f_c and high-frequency gain of this circuit?
Hint: It might be helpful to rearrange the expression above to make it conform with (2).

(b) Draw a Bode plot for the gain $\frac{V_{\text{out}}}{V_{\text{in}}}$ of this circuit.

(c) What type of filter is this?

Derivation of asymptotes

This is an supplement to the handout on filters and Bode magnitude plots.

Low-pass filter

$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{A}{1 + jBf}.$$

Low-frequency asymptote. As $f \to 0$, we have $1 \gg Bf$, so $1 + jBf \approx 1$. Then we have

$$20 \log \left| \frac{V_{\text{out}}}{V_{\text{in}}} \right| = 20 \log \left| \frac{A}{1 + jBf} \right| \approx 20 \log \left| \frac{A}{1} \right| = 20 \log |A|,$$

that is, at low frequencies, the gain converges[2] to the DC gain, whose magnitude is $|A|$.

High-frequency asymptote. As $f \to \infty$, $1 \ll Bf$, so $1 + jBf \approx jBf$. Then we have

$$20 \log \left| \frac{V_{\text{out}}}{V_{\text{in}}} \right| = 20 \log \left| \frac{A}{1 + jBf} \right| \approx 20 \log \left| \frac{A}{jBf} \right| = 20 \log \left| \frac{A}{Bf} \right| = 20 \log \left| \frac{A}{B} \right| - 20 \log f,$$

that is, the high-frequency asymptote has a slope of $-20\,\text{dB/decade}$ (the coefficient of $\log f$).

Corner frequency. The asymptotes intersect where (assuming B is real and positive),

$$20 \log |A| = 20 \log \left| \frac{A}{Bf_c} \right| = 20 \log |A| - 20 \log Bf_c,$$

and solving this for f_c yields $f_c = \frac{1}{B}$.

High-pass filter

$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{Af}{1 + jBf}.$$

Low-frequency asymptote. As $f \to 0$, we have $1 \gg Bf$, so $1 + jBf \approx 1$. Then we have

$$20 \log \left| \frac{V_{\text{out}}}{V_{\text{in}}} \right| = 20 \log \left| \frac{Af}{1 + jBf} \right| \approx 20 \log \left| \frac{Af}{1} \right| = 20 \log |A| + 20 \log f,$$

that is, the low-frequency asymptote has a slope of $20\,\text{dB/decade}$ (the coefficient of $\log f$).

High-frequency asymptote. As $f \to \infty$, $1 \ll Bf$, so $1 + jBf \approx jBf$. Then we have

$$20 \log \left| \frac{V_{\text{out}}}{V_{\text{in}}} \right| = 20 \log \left| \frac{Af}{1 + jBf} \right| \approx 20 \log \left| \frac{Af}{jBf} \right| = 20 \log \left| \frac{A}{jB} \right| = 20 \log \left| \frac{A}{B} \right|,$$

that is, at high frequencies, the gain converges to the high-frequency gain, whose magnitude is $20 \log \left| \frac{A}{B} \right|$.

Corner frequency. The asymptotes intersect where (assuming B is real and positive),

$$20 \log |Af_c| = 20 \log \left| \frac{A}{B} \right|,$$

and solving this for f_c yields $f_c = \frac{1}{B}$.

[2] Strictly speaking, our proof just shows that at lower frequencies, the gain is *approximately* the DC gain, not that it *converges to* the DC gain. Nonetheless, the gain does in fact converge to the DC gain, and this isn't too hard to prove (though we don't require you to do so). A similar comment applies for every other place where the gain is approximately a constant.

Band-pass filter

$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{Af}{(1 + jBf)(1 + jCf)}.$$

Analyzing this transfer function uses very similar ideas, there are just more terms to deal with. Here's a guide. Filling in the gaps (in the same spirit as the low-pass and high-pass filters above) is left as an exercise.

Low-frequency asymptote. For the low-frequency asymptote, take $f \to 0$ as usual, so that $1 \gg Bf$ and $1 \gg Cf$.

High-frequency asymptote. For the high-frequency asymptote, take $f \to \infty$ as usual, so that $1 \ll Bf$ and $1 \ll Cf$.

Horizontal line in the middle. For this function, though, there is a "middle line", which isn't *really* an asymptote.[3] This is nonetheless still part of the *straight-line approximation.* To find this middle approximation, assume that f is in such a range such 1 dominates in *one* of the denominator terms, and the f term dominates in the other. For example, if $B < C$, assume that $Bf \ll 1 \ll Cf$. (This might not be a very good approximation, but it's still insightful.) Your analysis should then yield a line that is horizontal on the Bode magnitude plot; that is, the passband of the filter.

Corner frequencies. It then follows that there are *two* corner frequencies: one for the intersection between the low-frequency asymptote and the horizontal line, and one for the intersection between the horizontal line and the high-frequency asymptote.

[3] In order to qualify as an asymptote, the function has to get arbitrarily close to it.