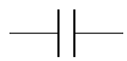
Circuits in the frequency domain

ENGR 40M lecture notes — August 2, 2017
Chuan-Zheng Lee, Stanford University

Our study of capacitors and inductors has so far been in the time domain. In some contexts, like transient response, this works fine, but in many others, the time domain can be both cumbersome and uninsightful. As we hinted last lecture, the frequency domain can give us a more powerful view of how circuits operate.

Quick reference

			
Impedance	$Z_C = \dfrac{1}{j2\pi fC}$	$Z_L = j2\pi fL$	$Z_R = R$
At DC, looks like	open circuit	short circuit	resistor
At very high frequencies, looks like	short circuit	open circuit	resistor

Some preliminary observations

Recall that, in a capacitor, $i = C\frac{dv}{dt}$. What happens if the voltage across the capacitor happens to be sinusoidal with amplitude V and frequency f, that is, with $v(t) = V\sin(2\pi ft + \phi)$? We would then have

$$i(t) = C\frac{dv}{dt} = 2\pi fCV\cos(2\pi ft + \phi) = \underbrace{2\pi fCV}_{I}\sin\left(2\pi ft + \phi + \frac{\pi}{2}\right).$$

That is, the current is also sinusoidal *with the same frequency*, a $\frac{\pi}{2}$ phase shift, and an amplitude of $2\pi fCV$. Ignoring phase, we might define a quantity, the ratio between the *amplitude* of voltage V and *amplitude* of current I,

$$\frac{V}{I} = \frac{V}{2\pi fCV} = \frac{1}{2\pi fC}.$$

Similarly, for an inductor, we can show that if $i(t) = I\sin(2\pi ft + \phi)$, then

$$v(t) = V\sin\left(2\pi ft + \phi + \frac{\pi}{2}\right), \qquad \text{where} \qquad \frac{V}{I} = 2\pi fL.$$

Finally, for a resistor, we can show that we would have $\frac{V}{I} = R$.

Impedance

Inspired by this observation, we define the **impedance** of a capacitor Z_C, of an inductor Z_L and of a resistor Z_R to be

$$Z_C = \frac{1}{j2\pi fC}, \qquad Z_L = j2\pi fL, \qquad Z_R = R.$$

You'll have noticed that a j mysteriously appeared in there. The salient thing to know about this is that it represents the *phase change* (by $\frac{\pi}{2}$) we brushed over above.[1]

[1] For those who have studied complex numbers, this relates to its argument.

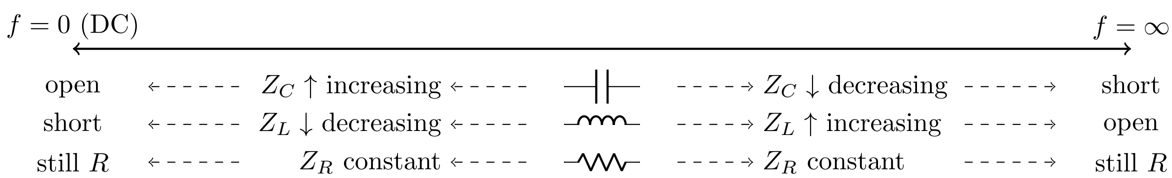
More precisely, j is defined to be the number such that $j^2 = -1$, and is known as the *imaginary unit*. You might have seen this in mathematics, where it took the symbol i. You probably thought that such imaginary ideas would never have *real* applications. As it happens, electrical engineers have very elegant uses for it.[2]

In this class, we will write j in our expressions for completion, but when performing precise calculations, we'll just pretend it's not there. Naturally, this will lead to answers that are not quite right.[3] For our purposes, this doesn't make much difference. (Fair warning: in some other applications, it makes a *huge* difference.)

Extreme cases

The special case $f = 0$ indicates how the circuit responds to the *DC component* of a Fourier series. We say that this is the circuit's behavior *at DC*. In this case, $Z_C = \infty$, so a capacitor looks like an open circuit; and $Z_L = 0$, so an inductor looks like a short circuit.

The opposite extreme is when $f \to \infty$. This isn't physically realizable, but it provides an intuition for how circuits will behave at *very high frequencies*. In this case, $Z_C \to 0$, so a capacitor looks like a short circuit, and $Z_L \to \infty$, so an inductor looks like an open circuit.



Note that our DC characterizations match the steady state from last week. This isn't a coincidence; in fact, the "steady state" we discussed is more accurately called the *DC steady state* (in contrast to AC).

Resistors don't exhibit frequency-dependent behavior. They just stay with $Z_R = R$, always. For this reason, we often don't bother replacing R with Z_R in our algebraic work.

Circuits in the frequency domain

Armed with our new tool, we can proceed to analyze circuits with sinusoidal sources, with no derivatives in sight—see Examples 1 and 2. In fact, because impedance represents a ratio between voltage and current, in the frequency domain, we can use impedance to analyze circuits *as if they were a resistor network*. The only difference is that these impedances can be *frequency-dependent*.

But there's still more. Because the frequency domain is just a means of expressing a signal as a *sum of sinusoids*, we can use a superposition-based argument to see that circuits just operate on each *frequency component* of an input signal independently. That is, any voltage or current in the circuit can be found by (1) decomposing the input into its frequency components, (2) applying our impedance-based analysis to each frequency component, and then (3) adding the results together.

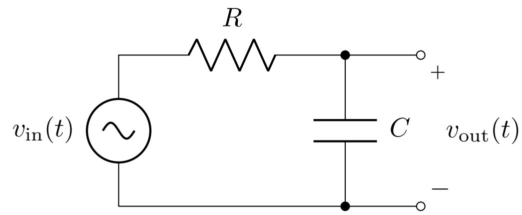
In many circuits, the output amplitude V_{out} is just a (frequency-dependent) multiple of the input amplitude V_{in}. In such cases, we often talk of the **gain** of a circuit, $\frac{V_{\text{out}}}{V_{\text{in}}}$. Where the gain is a function of frequency, we sometimes call it the *transfer function*.

[2] We don't have the mathematical tools to derive our expressions for impedance more rigorously. The "proper" way to do it involves the Fourier transform (an extension of the Fourier series) and complex exponentials (e^{jx}), and requires us to introduce a notion of "negative frequency".

[3] To get the precise answer, you would need to find the magnitude of a complex number. You're welcome to do this if you like; we'll accept either answer in homework and exams.

Examples

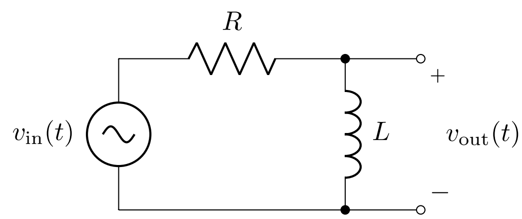
Example 1. Consider the circuit below, where $v_{\text{in}}(t)$ is a sinusoid with frequency f and amplitude V_{in}.



(a) Find an expression for V_{out}, the amplitude of $v_{\text{out}}(t)$, in terms of V_{in} and f.

(b) If $v_{\text{in}}(t)$ is a 1 kHz sinusoid with amplitude 300 mV, and $R = 10\,\text{k}\Omega$ and $C = 100\,\text{nF}$, what is the amplitude of $v_{\text{out}}(t)$?

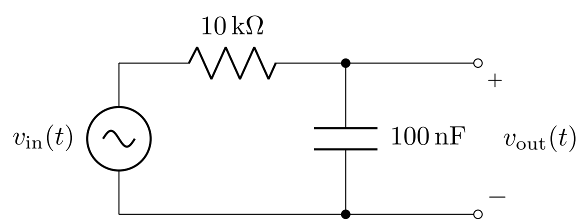
Example 2. Consider the circuit below, where $v_{\text{in}}(t)$ is a sinusoid with frequency f and amplitude V_{in}.



(a) Find an expression for V_{out}, the amplitude of $v_{\text{out}}(t)$, in terms of V_{in} and f.

(b) If $v_{\text{in}}(t)$ is a 700 Hz sinusoid with amplitude 1 V, and $R = 100\,\Omega$ and $L = 400\,\mu\text{H}$, what is the amplitude of $v_{\text{out}}(t)$?

Example 3. Consider the circuit from Example 1 (again).

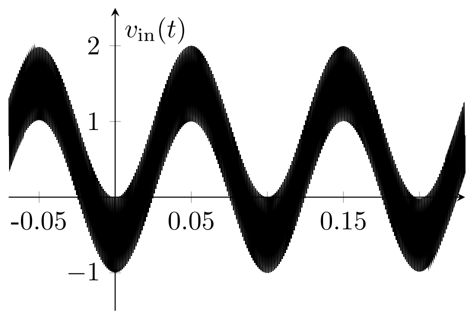


(a) Find the gain $\frac{V_{\text{out}}}{V_{\text{in}}}$ of the circuit, as a function of frequency.

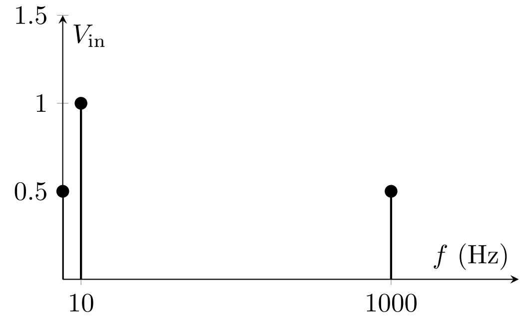
(b) What is the gain at (i) DC, and (ii) very high frequencies? Interpret this by describing what you would expect to see at $v_{\text{out}}(t)$.

(c) The time-domain and frequency-domain representations of an input $v_{\text{in}}(t)$ is shown below. Find the frequency-domain representation of the output $v_{\text{out}}(t)$.

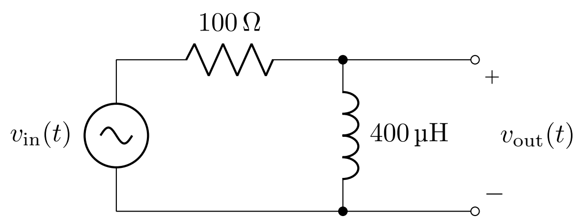
Time-domain representation



Frequency-domain representation



Example 4. Consider the circuit from Example 2 (again).



(a) Find the gain $\frac{V_{\text{out}}}{V_{\text{in}}}$ of the circuit, as a function of frequency.

(b) What is the gain at (i) DC, and (ii) very high frequencies? Interpret this by describing what you would expect to see at $v_{\text{out}}(t)$.

(c) The frequency-domain representation of an input $v_{\text{in}}(t)$ is shown below. Find the frequency-domain representation of the output $v_{\text{out}}(t)$, then (roughly) sketch its time-domain representation.

Frequency-domain representation of $v_{\text{in}}(t)$