Transient response of RC and RL circuits

ENGR 40M lecture notes — July 26, 2017

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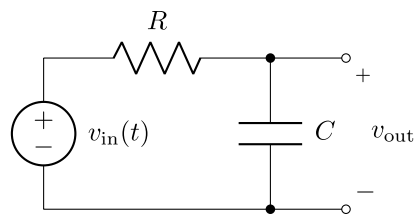
Resistor–capacitor (RC) and resistor–inductor (RL) circuits are the two types of *first-order* circuits: circuits either one capacitor or one inductor. In many applications, these circuits respond to a *sudden change* in an input: for example, a switch opening or closing, or a digital input switching from low to high. Just after the change, the capacitor or inductor takes some time to charge or discharge, and eventually settles on its new steady state. We call the response of a circuit immediately after a sudden change the *transient response*, in contrast to the steady state.

A first example

Consider the following circuit, whose voltage source provides $v_{\text{in}}(t) = 0$ for $t < 0$, and $v_{\text{in}}(t) = 10\,\text{V}$ for $t \geq 0$.



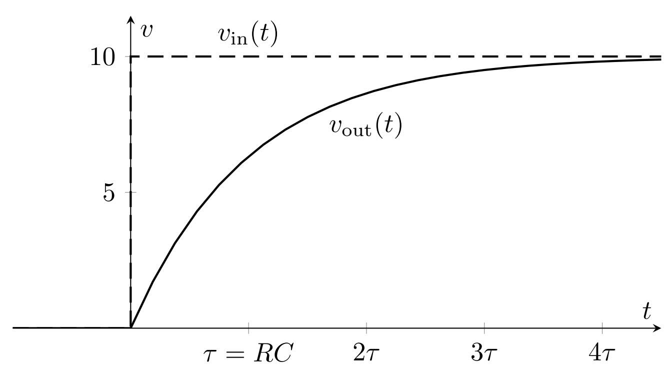
A few observations, using steady state analysis. *Just before* the step in v_{in} from $0\,\text{V}$ to $10\,\text{V}$ at $t = 0$, $v_{\text{out}}(0^-) = 0\,\text{V}$. Since v_{out} is across a capacitor, v_{out} *just after* the step must be the same: $v_{\text{out}}(0^+) = 0\,\text{V}$. *Long after* the step, if we wait long enough the circuit will reach steady state, then $v_{\text{out}}(\infty) = 10\,\text{V}$.

What happens in between? Using Kirchoff's current law applied at the top-right node, we could write

$$\frac{10\,\text{V} - v_{\text{out}}}{R} = C\frac{dv_{\text{out}}}{dt}.$$

Solving this differential equation, then applying the initial conditions we found above, would yield (in volts)

$$v_{\text{out}}(t) = 10 - 10e^{-\frac{t}{RC}}.$$



What's happening? Immediately after the step, the current flowing through the resistor—and hence the capacitor (by KCL)—is $i(0^+) = \frac{10\,\text{V}}{R}$. Since $i = C\frac{dv_{\text{out}}}{dt}$, this current causes v_{out} to start rising. This, in turn, *reduces* the current through the resistor (and capacitor), $\frac{10\,\text{V} - v_{\text{out}}}{R}$. Thus, the rate of change of v_{out} decreases as v_{out} increases. The voltage $v_{\text{out}}(t)$ technically never reaches steady state, but after about $3RC$, it's very close.

Transient response equation

It turns out that *all* first-order circuits respond to a sudden change in input with some sort of exponential decay, similar to the above. Therefore, we don't solve differential equations every time we see a capacitor or an inductor, and we won't ask you to solve any.

Instead, we use the following shortcut: In any first-order circuit, if there is a sudden change at $t = 0$, the transient response for a voltage is given by

$$v(t) = v(\infty) + [v(0^+) - v(\infty)]e^{-t/\tau},$$

where $v(\infty)$ is the (new) steady-state voltage; $v(0^+)$ is the voltage just *after* time $t = 0$; τ is the *time constant*, given by $\tau = RC$ for a capacitor or $\tau = L/R$ for an inductor, and in both cases R is the *resistance seen by the capacitor or inductor*.

The transient response for a current is the same, with $i(\cdot)$ instead of $v(\cdot)$:

$$i(t) = i(\infty) + [i(0^+) - i(\infty)]e^{-t/\tau}.$$

What do we mean by the "resistance seen by the capacitor/inductor"? Informally, it means the resistance you would think the rest of the circuit had, if you were the capacitor/inductor. More precisely, you find it using these steps:

1. Zero out all sources (*i.e.* short all voltage sources, open all current sources)
2. Remove the capacitor or inductor
3. Find the resistance of the resistor network whose terminals are where the capacitor/inductor was

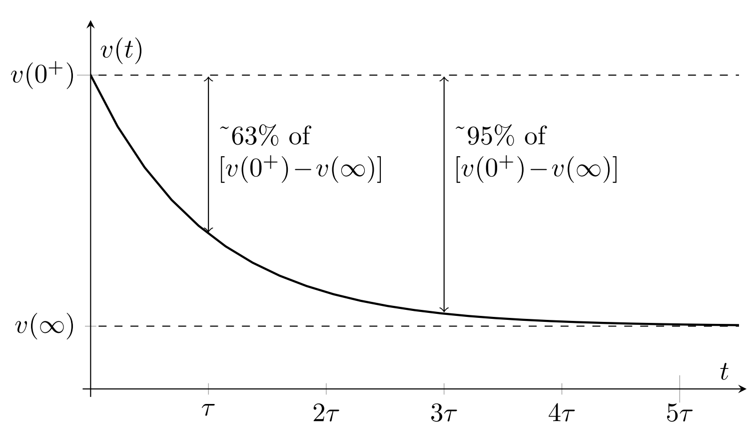
About the time constant

The time constant τ (the Greek letter *tau*) has units of seconds (verify, for both RC and R/L), and it governs the "speed" of the transient response. Circuits with higher τ take longer to get close to the new steady state. Circuits with short τ settle on their new steady state very quickly.

More precisely, every time constant τ, the circuit gets $1 - e^{-1} \approx 63\%$ of its way closer to its new steady state. Memorizing this fact can help you draw graphs involving exponential decays quickly.

After 3τ, the circuit will have gotten $1 - e^{-3} \approx 95\%$ of the way, and after 5τ, more than 99%. So, after a few time constants, for practical purposes, the circuit has reached steady state. Thus, the time constant is itself a good rough guide to "how long" the transient response will take.

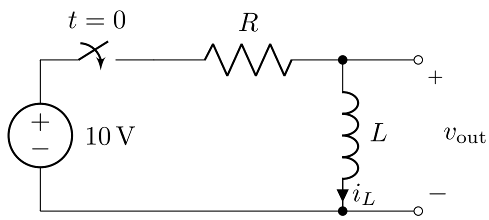
Of course, mathematically, the steady state is actually an asymptote: it never *truly* reaches steady state. But, unlike mathematicians, engineers don't sweat over such inconsequential details.



Examples and exercises

Example 1. Use the first-order transient response equation to verify the result for $v_{\text{out}}(t)$ that was found in the RC example on page 1.

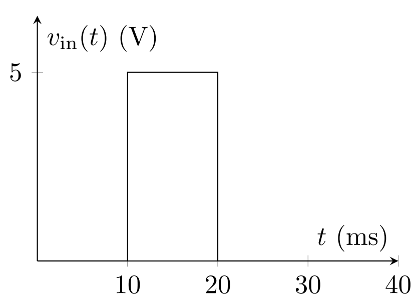
Exercise 1. *[Not examinable material.]* In the following circuit, the switch closes at time $t = 0$, before which it had been open for a long time. Use Kirchoff's voltage law to write a differential equation for the following circuit, and solve it to find $v_{\text{out}}(t)$. Verify that your answer matches what you would get from using the first-order transient response equation.

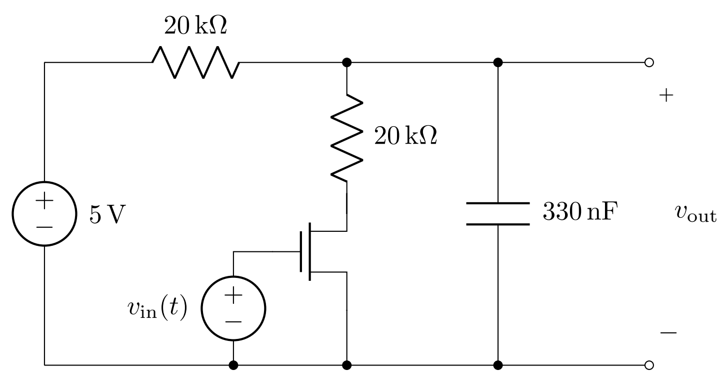


Example 2. In the above circuit (the same as for Exercise 1), the switch closes at time $t = 0$, before which it had been open for a long time.

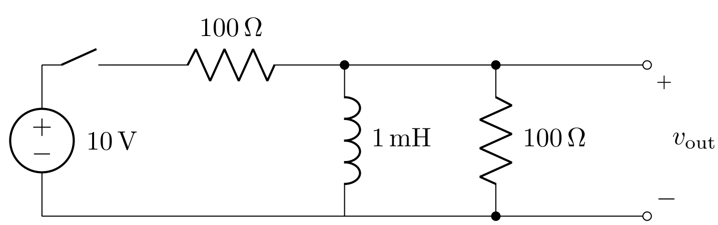
(a) Find and plot $i_L(t)$.
(b) Find and plot $v_{\text{out}}(t)$.

Example 3. In the below circuit, $v_{\text{in}}(t)$ is as is shown in the plot. (For all times outside the plot, $v_{\text{in}}(t) = 0\,\text{V}$.) Find and plot $v_{\text{out}}(t)$, for $t > 0$.



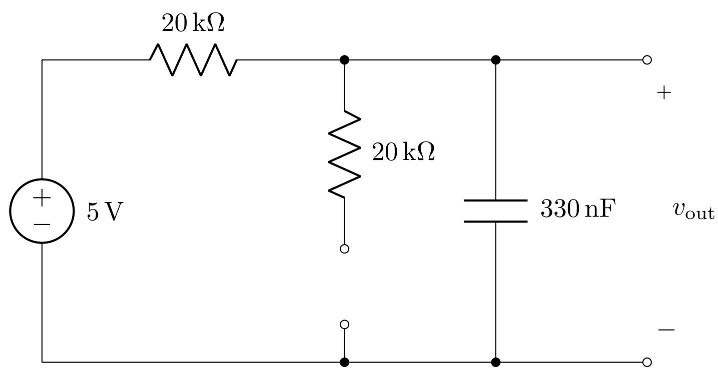


Example 4. In the below circuit, the switch has been closed for a long time before $t = 0$. The switch then *opens* at $t = 0$, and then *closes* at $t = 100\,\mu\text{s}$. Find and plot $v_{\text{out}}(t)$.

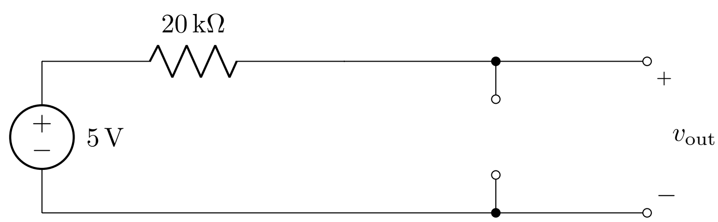


Solution to Example 3

<u>When $t < 10\,\text{ms}$.</u> When $v_{\text{in}}(t) = 0$, the NMOS transistor is off, so the circuit reduces to this:

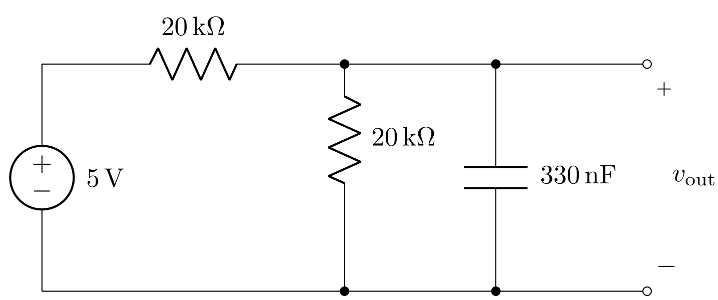


Since $v_{\text{in}}(t) = 0$ for all $t < 10\,\text{ms}$, the circuit is in steady state at $t = 10\,\text{ms}^-$. Furthermore, that $20\,\text{k}\Omega$ resistor on the right doesn't do anything while the transistor is off. So the circuit reduces to this:



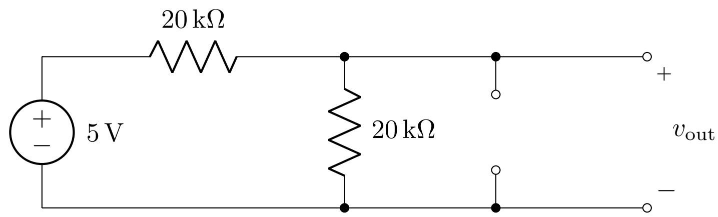
from which we see that $v_{\text{out}}(t) = 5\,\text{V}$ at steady state while the transistor is off, including when $t < 10\,\text{ms}$.

<u>When $10\,\text{ms} < t < 20\,\text{ms}$.</u> During this period, $v_{\text{in}}(t) = 5\,\text{V}$, so the transistor is on ($v_{GS} = 5\,\text{V} - 0\,\text{V} = 5\,\text{V}$). Then we have this circuit:



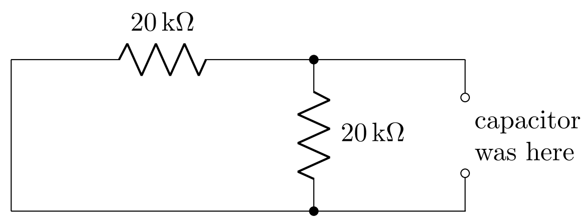
The voltage v_{out} is the voltage across a capacitor, which can't change instantaneously, so $v_{\text{out}}(10\,\text{ms}^+) = v_{\text{out}}(10\,\text{ms}^-) = 5\,\text{V}$. (Caution! The logic isn't that the *output voltage* can't change instantaneously, it's the the voltage *across the capacitor* can't.)

To find the $v(\infty)$ term of the equation, for this part, we find the steady state value of the circuit while the transistor is on—in other words, the value that $v_{\text{out}}(t)$ *would* converge to if the transistor stayed on forever. (The circuit can't predict the future, so it doesn't know that the transistor will turn off soon.) To find the steady state value, we replace the capacitor with an open circuit again:



Now we can see that there is just a voltage divider, and $v_{\text{out}}(\infty)$ in this state *would be* 2.5 V.

The time constant is $\tau = RC$, where R is the resistance seen by the capacitor. To find this, we short (zero) the voltage source and imagine measuring the resistance from the capacitor:



And now we can see that it's just two 20 kΩ resistors in parallel, yielding $R = 10\,\text{k}\Omega$. Then

$$\tau = RC = 10\,\text{k}\Omega \cdot 330\,\text{nF} = 3.3\,\text{ms}.$$

Putting it all together, for $10\,\text{ms} < t < 20\,\text{ms}$:

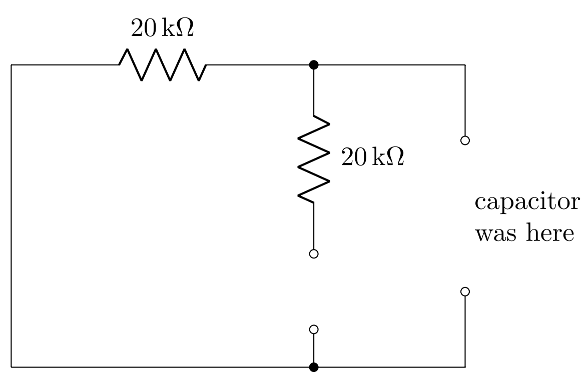
$$\begin{aligned} v_{\text{out}}(t) &= v_{\text{out}}(\infty) + [v_{\text{out}}(10\,\text{ms}^+) - v_{\text{out}}(\infty)]e^{-\frac{t-10\,\text{ms}}{\tau}} \\ &= 2.5 + [5 - 2.5]e^{-\frac{t-10\,\text{ms}}{3.3\,\text{ms}}} \qquad \text{(V)} \\ &= 2.5 + 2.5e^{-\frac{t-10\,\text{ms}}{3.3\,\text{ms}}} \qquad \text{(V)} \end{aligned}$$

<u>When $t > 20\,\text{ms}$.</u> Here, the transistor has turned off again. Because the voltage across the capacitor can't change instantaneously, $v_{\text{out}}(20\,\text{ms}^+) = v_{\text{out}}(20\,\text{ms}^-)$, and $v_{\text{out}}(20\,\text{ms}^-)$ falls into the equation we just found:

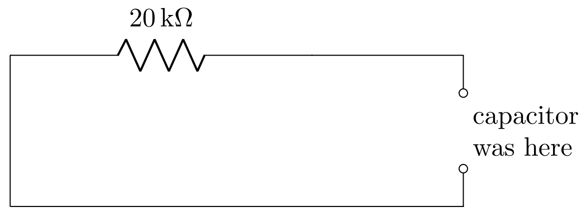
$$\begin{aligned} v_{\text{out}}(20\,\text{ms}^-) &= 2.5 + 2.5e^{-\frac{20\,\text{ms}-10\,\text{ms}}{3.3\,\text{ms}}} \\ &= 2.62\,\text{V}. \end{aligned}$$

The $v_{\text{out}}(\infty)$ term is just the steady state when the transistor is off, which we've already found above; it was 5 V.

As for the time constant, if we short the voltage source and remove the capacitor while the transistor is off, we get this:



That resistor on the right isn't really part of the circuit, so really we just have this:



And now it's clear that $R = 20\,\mathrm{k\Omega}$. Then

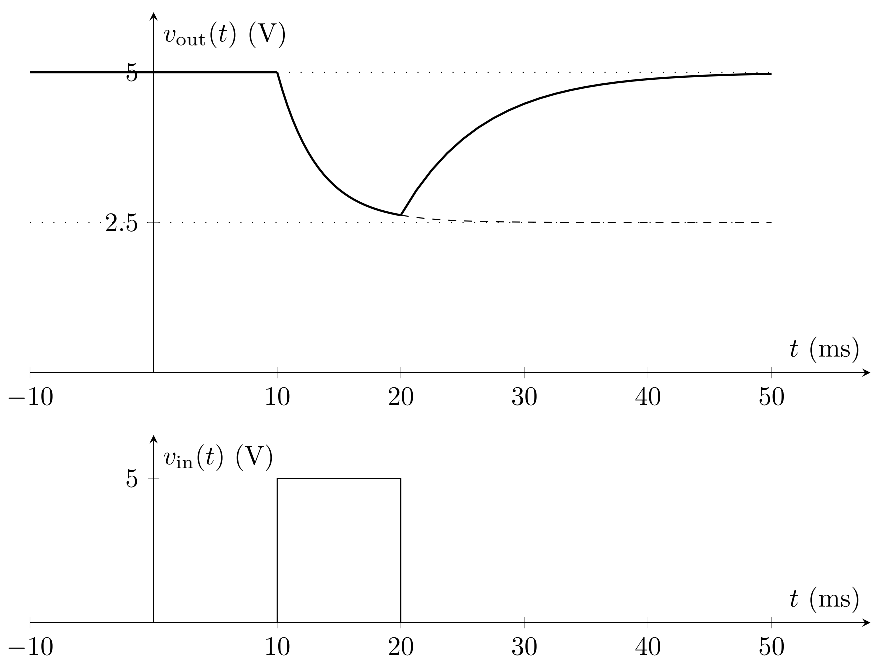
$$\tau = RC = 20\,\mathrm{k\Omega} \cdot 330\,\mathrm{nF} = 6.6\,\mathrm{ms}.$$

Note that this is twice as long as the time constant while the transistor was on.

Putting it all together, for $t > 20\,\mathrm{ms}$:

$$\begin{aligned} v_{\mathrm{out}}(t) &= v_{\mathrm{out}}(\infty) + [v_{\mathrm{out}}(20\,\mathrm{ms}^{+}) - v_{\mathrm{out}}(\infty)]e^{-\frac{t-20\,\mathrm{ms}}{\tau}} \\ &= 5 + [2.62 - 5]e^{-\frac{t-20\,\mathrm{ms}}{6.6\,\mathrm{ms}}} \qquad (\mathrm{V}) \\ &= 5 - 2.38e^{-\frac{t-10\,\mathrm{ms}}{6.6\,\mathrm{ms}}} \qquad (\mathrm{V}) \end{aligned}$$

Here is a plot, with $v_{\mathrm{in}}(t)$ drawn underneath for ease of reference. The dashed line is the curve that would have been followed if the transistor had been on forever.



You can see a simulation of this circuit in action at `http://everycircuit.com/circuit/5919486916165632`.