Math 21 - Spring 2015
Classnotes, Week 5

This week we will cover Sections 7.1 and 7.2, as well as some extra material which is covered in these notes.

5.1 An example of a differential equation: Bacterial growth

Once one knows about the idea of a rate of change, one starts realizing that many of the most important problems in science, when formulated mathematically, give rise to **differential equations** (we will define this expression later). Let's explore one such problem in more detail to see how this happens.

Suppose that you have an old jar of yogurt in the refrigerator, and it is growing bacteria. For this problem, we will let P (for population) denote the number of bacteria in the jar of yogurt. Now, the number of bacteria changes with time, so P is a function of t, time. As you keep an eye on your jar of yogurt over time, you do not know much about the function P at first. In other words, you might not be able to write down a formula for the rule of the function $P\colon [0,\infty) \to \mathbb{R}$.

Despite this, maybe you notice that the situation is quickly getting out of control: the more bacteria there is in the yogurt, the more bacteria are "born" and added to the population when you look the next day. When you think about it, it makes sense: After looking it up on Wikipedia, you know that bacteria reproduce by binary fission – the bacterium literally just splits in two and ends up being two bacteria. If for example each bacterium splits in two once every day, then the next day when you look there would be twice as many bacteria in the yogurt jar. Now you understand bacteria a little bit better, but you still can't write down a rule for $P(t)$.

Upon thinking about it more though you might realize that the fact that the bacteria are reproducing by splitting in two means that the *rate of growth of the population of bacteria is proportional to the size of the population.* What does that mean?

Well, first, we say that a quantity y is proportional to a quantity x if x and y are related by an equation

$$y = kx,$$

for some number $k \neq 0$ called the **constant of proportionality**.

For example, the price y of x boxes of cereal is proportional to the number of boxes of cereal that you buy. If each box costs \$2, then if you buy x boxes of cereal you will pay

$$y = 2x$$

dollars. (So here, the constant k is simply the cost of one box of cereal.) One important thing to notice here is that if you buy 5 times as many boxes of cereal, it will cost you 5 times as much money. (There is no bulk discount, and cereal does not become more expensive as it becomes scarce because you have bought too much.)

Using this concept, the English sentence "the rate of growth of the population of bacteria is proportional to the size of the population" translates to math as

$$\frac{dP}{dt}, \qquad \text{(the rate of growth of the population, or number of bacteria "born" each day)}$$

is proportional to

$$P(t), \qquad \text{(the size of the population at time } t\text{);}$$

which is the equation

$$\frac{dP}{dt} = kP(t).$$

for some constant $k \neq 0$.

Suppose for a moment that $k = 1$. Then the equation is:

$$\frac{dP}{dt} = P(t).$$

This is still not a rule for $P(t)$! However, it is a **differential equation** which is satisfied by the function P. If we can just find one (or many!) function P whose rule satisfies this equation, then we will know a (possible) rule for P. You might remember that a function that is equal to its derivative is $P\colon [0, \infty) \to \mathbb{R}$, $P(t) = e^t$.

Let's go back to the more general

$$\frac{dP}{dt} = kP(t).$$

Simply by thinking about it, we might find that a solution to this differential equation is $P\colon [0, \infty) \to \mathbb{R}$, $P(t) = e^{kt}$. (Check it!)

By knowing one fact about bacteria (that they reproduce using binary fission) we were able to determine that if $P(t)$ is the number of bacteria in your jar of yogurt at time t, then a possible guess for the rule for P is $P(t) = e^{kt}$, for some k.

5.2 Finding k

To know everything about our population of bacteria, we need to compute the value of the constant k. This can be done by observation and computation.

Suppose that the bacteria in the jar of yogurt in the refrigerator undergo binary fission once a day. Then if you count the bacteria in the morning on day, and then again the next morning, there will be twice as many bacteria. (For example if there are 10 bacteria on the morning of day 0, and during the day each of them splits into two bacteria, on the morning of the next day, day 1, there will be 20 bacteria.)

More generally we can say that $P(1)$, the number of bacteria on day 1, is twice $P(0)$, the number of bacteria on day 0. In math, this translates as

$$P(1) = 2P(0).$$

The solution we found was

$$P(t) = e^{kt},$$

so we need a number k such that

$$P(1) = e^k = 2 = 2e^0 = 2P(0).$$

This number is

$$k = \ln 2.$$

Therefore a possible rule for the function P that denotes the number of bacteria in your refrigerator at time t is

$$P(t) = e^{t \ln 2} = 2^t.$$

It is easy to imagine that under other circumstances (maybe if the jar was left on the counter rather than in the refrigerator) then maybe the bacteria would split twice during the day. (In that case, on a day when we counted 10 bacteria on the morning of day 0, we would count 40 bacteria the next morning on day 1.)

So this time

$$P(1) = 4P(0).$$

Therefore we solve

$$P(1) = e^k = 4 = 4e^0 = 4P(0).$$

This equation has solution $k = \ln 4$. Therefore, when the jar is left on the counter, a possible rule for the function P giving the population at time t is

$$P(t) = 4^t.$$

The take away point from this whole story is that by simply observing a jar of yogurt, and thinking about how bacteria might reproduce, and writing our observations down in mathematical language, we can come up relatively quickly with a (possible) rule for the function P. This point of view has proved incredibly fruitful for us to make sense of the world. By writing down differential equations and solving them, human beings have been able to write down functions that make sense of the world around them and allow them to make predictions about what will happen in the future.

The book discusses population growth at the beginning of Section 7.1, and introduces two models for population growth. For another example of modeling phenomena using differential equations (this is what we call what we have just done: when we write down an equation that describes something that we have observed, we call it “modeling”) you can also read the subsection on “A Model for the Motion of a Spring.”

5.3 Vocabulary

Since we will be talking about a bunch of new things, we will need a bunch of new words to describe these things. You are responsible for all of the definitions below.

A **differential equation** is an equation (i.e. something with a "=" and things on both sides) that contains an unknown function and one or more of its derivatives. The **order** of the differential equation is the order of the highest derivative that occurs in the equation. For example, if y is a function of x, then

$$y''' + \cos(y)xy' = \frac{1}{xy}$$

is a differential equation, since it is an equation that relates the unknown function y to its derivatives y' and y'''. Furthermore, it is a third-order differential equation, since the third derivative y''' appears, but no derivative of higher order appears.

A **solution** of a differential equation is a function that satisfies the differential equation when the function and its derivatives are substituted into the equation. For example, we said that $P\colon [0,\infty) \to \mathbb{R}$ given by $P(t) = e^{t\ln 2}$ is a solution to the first-order differential equation

$$\frac{dP}{dt} = \ln 2\, P(t)$$

since taking the derivative of P we have

$$\frac{dP}{dt} = \ln 2\, e^{t\ln 2},$$

and plugging in $P(t) = e^{t\ln 2}$ and $\frac{dP}{dt} = \ln 2\, e^{t\ln 2}$ gives

$$\ln 2\, e^{t\ln 2} = \ln 2\, e^{t\ln 2},$$

which is true.

When we ask you to **solve** a differential equation, we are asking you to find **all** of the solutions of a given differential equation. So if you are asked to solve the first-order differential equation

$$\frac{dP}{dt} = \ln 2\, P(t)$$

you have to ask yourself, is $P\colon [0,\infty) \to \mathbb{R}$ given by $P(t) = e^{t\ln 2}$ the *only* solution?

It turns out that it is not. For any constant C, the function $P\colon [0,\infty) \to \mathbb{R}$ given by $P(t) = Ce^{t\ln 2}$ is also a solution of the differential equation (check that this is true!). This should remind you of when we took integrals and had to put in the "$+\,C$". In fact, taking an integral was exactly solving a simple differential equation: The integral

$$\int 4x^3\, dx$$

is exactly asking to solve the first-order differential equation

$$y' = 4x^3.$$

All of the solutions of this differential equation are $y\colon \mathbb{R} \to \mathbb{R}$ given by the rule $y(x) = x^4 + C$, for C an arbitrary constant. We call the family of solutions (the expression with the constant C in it) the **general solution** of the differential equation.

One can solve for the constant C if one has an additional piece of information. For example, when considering the population of bacteria growing in the refrigerator, for the first-order differential equation

$$\frac{dP}{dt} = \ln 2\, P(t)$$

we obtained the general solution $P\colon [0,\infty) \to \mathbb{R}$, $P(t) = Ce^{t\ln 2}$. If one knows in addition that at time $t = 0$ there were 4 bacteria in the jar, then one can solve for C:

$$P(0) = Ce^0 = C = 4.$$

Therefore the exact rule for the function P that describes the population of bacteria in the refrigerator at time t is

$$P(t) = 4e^{t\ln 2} = 2^2\, 2^t = 2^{t+2}.$$

The additional pieces of information given along with a differential equation are called **initial conditions**. The problem of finding a solution to a differential equation that also satisfies the initial conditions is called an **initial value problem**.

To recap: A differential equation by itself can be solved by giving a general solution (or many), which will typically have some arbitrary constants in it. Once we add in some initial conditions, the problem is now called an initial value problem, and (if we are lucky) the initial values allow us to solve for the constants to get a unique solution.

If you would like to read more background about differential equations, the Introduction to Boyce and DiPrima's excellent *Elementary Differential Equations and Boundary Value Problems* is posted under "Resources," and offers a beautiful and more general discussion of these ideas.

5.4 Direction fields

In Weeks 5 and 6, we will cover only first-order differential equations, and in these notes we will let y be a function of x unless otherwise specified. Even more specifically, we will cover first-order differential equations of the form

$$y' = F(x, y),$$

for some function F of x and y. In other words, the first derivative of y will depend on the function y and the independent variable x explicitly. An example of such a differential equation is

$$y' = e^x y + y^2 \sin x.$$

We briefly note that not all first-order differential equations can be written in this form. For example, the first-order differential equation

$$(y')^2 = x$$

cannot be solved uniquely for y'. If we tried to solve for y', we would get

$$y' = \sqrt{x}$$

and

$$y' = -\sqrt{x}$$

This is because the function $f\colon \mathbb{R} \to \mathbb{R}$ given by the rule $f(x) = x^2$ is not one-to-one, and therefore does not have an inverse. We will assume that this never happens to us, although if it did happen that wouldn't be so bad.

Given an differential equation

$$y' = F(x, y),$$

a first step in figuring out what is happening is to draw (or have a computer draw for you) a **direction field**. The book does a really good job of discussing direction fields in Section 7.2, and you should read it.

5.5 Euler's method

For an initial value problem

$$y' = F(x, y), \qquad \text{and} \qquad y(x_0) = y_0$$

a powerful way to *estimate* a solution is Euler's method. This is also covered in the book in Section 7.2, and you should read about it.

5.6 Autonomous equations

An important kind of first-order differential equation is one of the form

$$y' = f(y).$$

In other words, the first derivative of y depends only on y (and not on x). An example of such a differential equation is

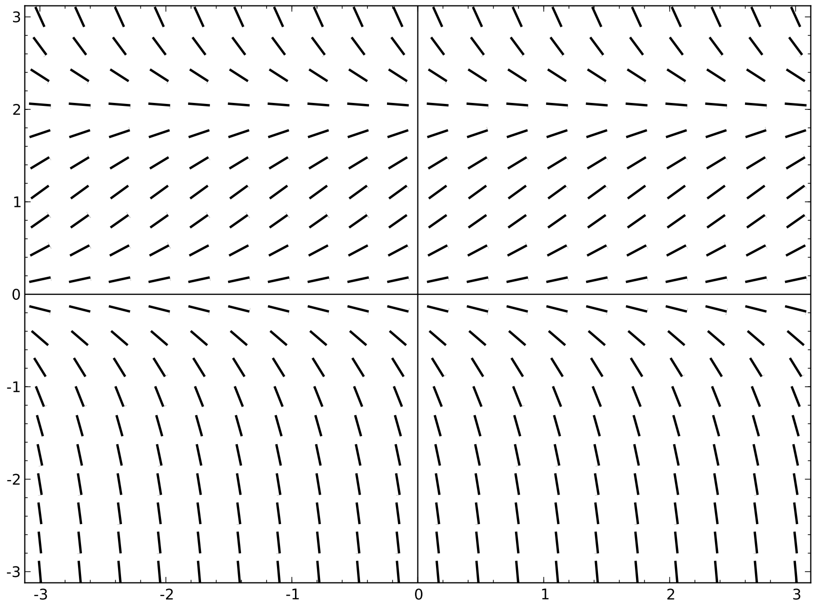
$$y' = y,$$

which we have encountered before. Such a differential equation is called an **autonomous differential equation**.

We study these equations in particular because they are relatively easy to make sense of. Because they are relatively simple, geometric methods (i.e. pictures of the direction field) can give us important information about the solutions of the differential equation. To get some ideas about what kind of information we can get, let us consider a few direction fields.

Below is the direction field for the first-order autonomous differential equation

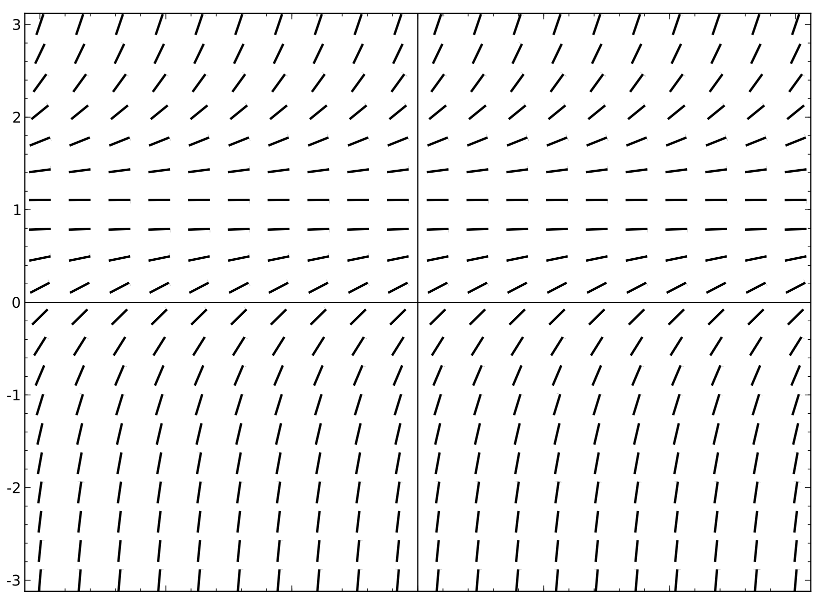
$$y' = -y^2 + 2y$$



Notice that when $y = 0$ and $y = 2$ the slopes are flat. This is because when $y = 0$ or $y = 2$ and we plug this in on the right-hand side, we get $y' = 0$. You should also notice that solutions that start below $y = 0$ all go down, solutions that start between $y = 0$ and $y = 2$ all go up towards $y = 2$, and solutions that start above $y = 2$ all go down towards $y = 2$.

Consider now the direction field for the first-order autonomous differential equation

$$y' = (y - 1)^2.$$



Notice that when $y = 1$ the slopes are flat, which again makes sense since plugging in $y = 1$ on the right-hand side, we get $y' = 0$. Further notice that solutions that start below $y = 1$ all go up towards $y = 1$, and solutions that start above $y = 1$ all go up towards ∞.

Given a first-order autonomous differential equation

$$y' = f(y),$$

the values of y such that $f(y) = 0$ are called the **critical points** of the differential equation. If y_0 is a critical point of the differential equation, then the constant solution $y(x) = y_0$ is called an **equilibrium solution** of the differential equation.

If $y = y_0$ is a critical point of the differential equation, we have seen that other solutions can either go towards the equilibrium solution $y(x) = y_0$, move away from the equilibrium solution $y(x) = y_0$, or do one or the other depending on which side of the equilibrium solution we begin. When solutions tend to go towards the equilibrium solution $y(x) = y_0$, we say that this solution is a **stable equilibrium solution** or the critical point y_0 is a **stable critical point**. When solutions tend to go away from the equilibrium solution $y(x) = y_0$, we say that this solution is an **unstable equilibrium solution** or the critical point y_0 is an **unstable critical point**. In the third case, we say that $y(x) = y_0$ is a **semi-stable equilibrium solution** or the critical point y_0 is a **semi-stable critical point**.

We can determine whether a critical point is stable, unstable, or semi-stable by using a sign chart for y'. We show how for the two examples discussed above:

Example 1:
Consider again the first-order autonomous differential equation

$$y' = -y^2 + 2y.$$

The polynomial $f(y) = -y^2 + 2y$ factors as $f(y) = -y(y - 2)$, and therefore has two roots, $y = 0$ and $y = 2$. These two values of y are the critical points. Remembering that this is an expression for y', we can make a sign chart:

y		0		2	
$f(y)$	$-$		$+$		$-$
inc/dec	$\swarrow$		$\nearrow$		$\swarrow$

From the sign chart, we see that solutions move away from $y = 0$ when $y < 0$, and solutions move away from $y = 0$ when $y > 0$. Therefore $y = 0$ is an unstable critical point, or the constant solution $y(x) = 0$ is an unstable equilibrium solution. On the other hand, solutions move towards $y = 2$ when $0 < y < 2$, and solutions also move towards $y = 2$ when $y > 2$. Therefore $y = 2$ is a stable critical point, or the constant solution $y(x) = 2$ is a stable equilibrium solution.

Example 2:
Consider now the first-order autonomous differential equation

$$y' = (y - 1)^2.$$

The polynomial $f(y) = (y-1)^2$ has one root, $y = 1$, and this is the only critical point. We again make a sign chart:

y		1	
$f(y)$	$+$		$+$
inc/dec	$\nearrow$		$\nearrow$

From the sign chart, we see that solutions move towards $y = 1$ when $y < 1$, and solutions move away from $y = 1$ when $y > 1$. Therefore $y = 1$ is a semi-stable critical point, or the constant solution $y(x) = 1$ is a semi-stable equilibrium solution.

5.7 More population dynamics

Throughout this section, let t denote time, and let P denote the size of a given population at time t. We will only consider $t > 0$ and $P(t) > 0$, since neither negative time nor negative populations make sense in the real world.

Example 1: Consider again the differential equation

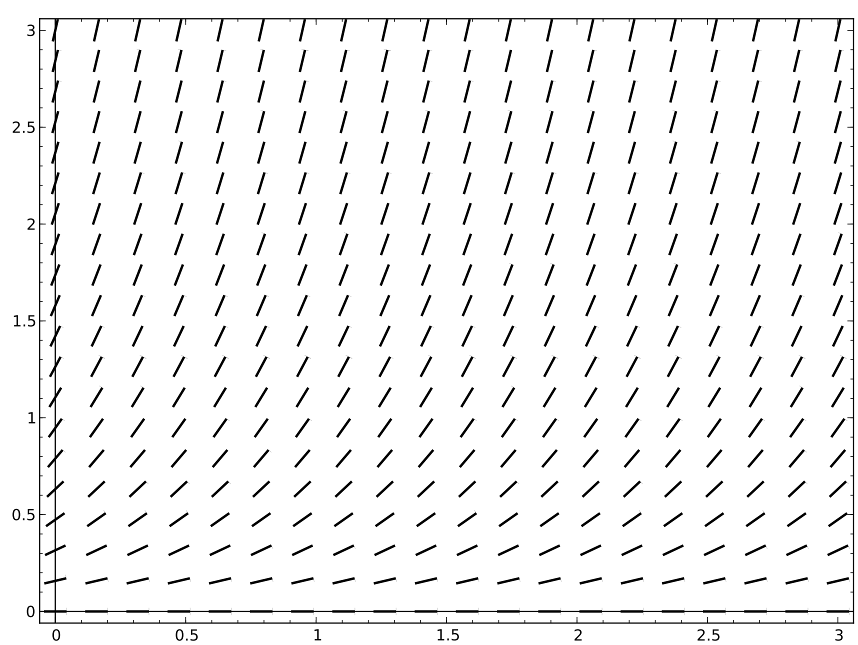
$$\frac{dP}{dt} = kP(t),$$

for some positive constant k. This model is called the **exponential growth model**, and in this context the constant k is called the **rate of growth**.

For example, if we let $k = 2$, we are considering the differential equation

$$\frac{dP}{dt} = 2P(t).$$

This differential equation has direction field



We now discuss the equilibrium solutions. Here the right-hand side is $2P$, so the only critical point is $P = 0$. The sign chart looks like:

P	0
$2P$	$+$
inc/dec	$\nearrow$

From this we see that the critical point $P = 0$ is unstable.

The biological interpretation of our findings is the following: Suppose that we have a population that has the property that as long as there is at least one individual, the population will increase without bounds. (This happens to populations that live under ideal conditions.) Then this population can be modeled using the exponential growth model.

In mathematical words, we would describe this model by saying that for any initial value $P(t_0) = P_0 > 0$, the population increases without bounds.

Example 2: Consider now the differential equation

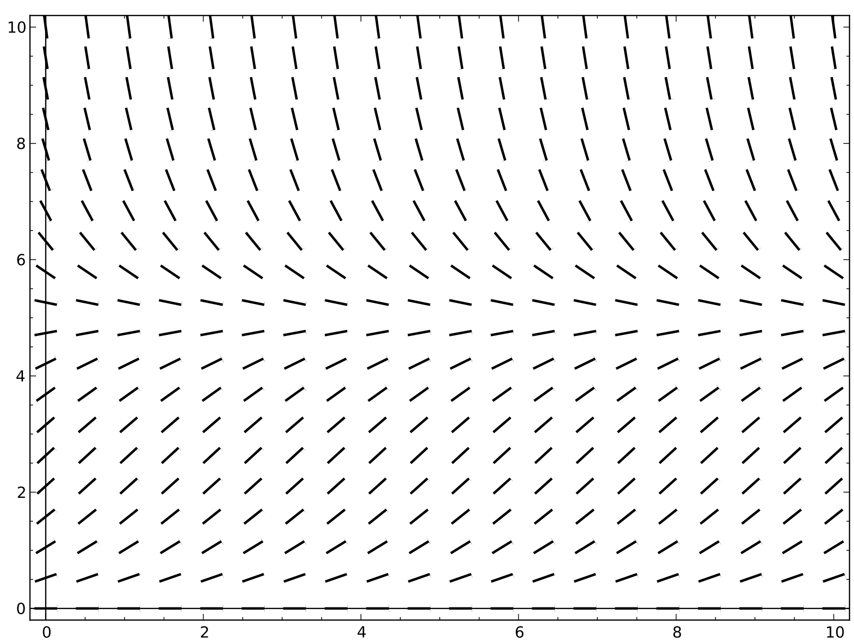
$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right),$$

for some positive constants k and M. This model is called the **logistic growth model**. In this context the constant k is called the **intrinsic growth rate**, and the constant M is called the **carrying capacity**.

For example, if we let $k = 1$ and $M = 5$, we are considering the differential equation

$$\frac{dP}{dt} = P\left(1 - \frac{P}{5}\right).$$

This differential equation has direction field



We now investigate the equilibrium solutions. Here the right-hand side is $P(1-\frac{P}{5})$, so the critical points are $P=0$ and $P=5$. The sign chart looks like:

P	0		5	
$P\left(1-\frac{P}{5}\right)$		+		−
inc/dec		↗		↙

From this we see that the critical point $P=0$ is unstable, and the critical point $P=5$ is stable.

The biological interpretation of our findings is the following: Suppose that we have a population which we believe will stabilize at 5 individuals in the long run. (This happens to populations that live in an environment that can only support so many individuals, because of limits in the food supply for example.) Then this population can be modeled by the logistic growth model with carrying capacity $M=5$.

In mathematical words, we can say that if we have an initial value $P(t_0)=P_0>0$ then the population will approach 5 as $t\to\infty$. (If $P_0=5$, then the population remains at 5 forever.)

Example 3: Consider finally the differential equation

$$\frac{dP}{dt}=-kP\left(1-\frac{P}{m}\right),$$

for some positive constants k and m. This model is called the **threshold growth model**. In this context the constant k is again called the **intrinsic growth rate**, and the constant m is called the **threshold level**. Notice that this model differs from the logistic growth model only in the fact that there is a minus sign in front of the right-hand side.

For example, if we let $k=1$ and $m=5$, we are considering the differential equation

$$\frac{dP}{dt}=-P\left(1-\frac{P}{5}\right).$$

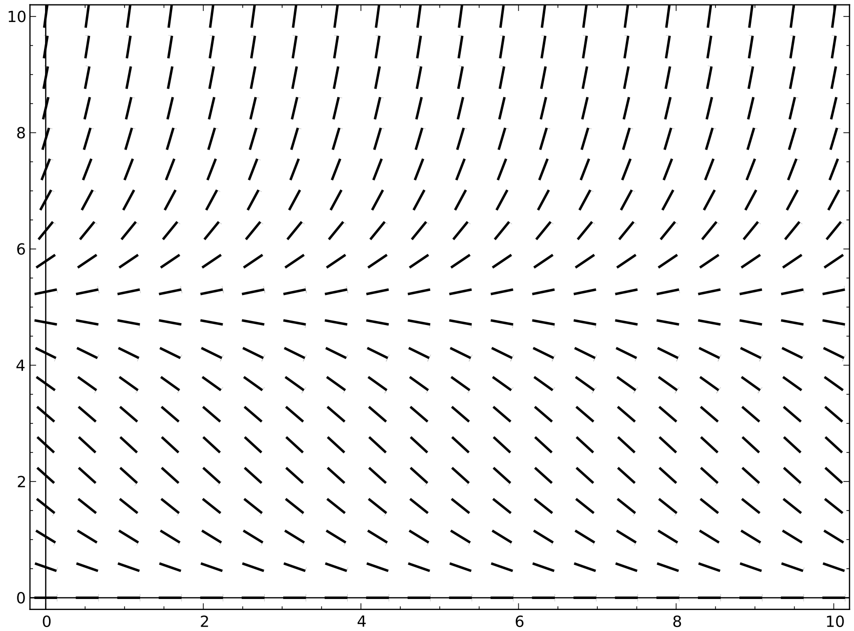
(The direction field for this example is on the next page.)

We find the equilibrium solutions. Here the right-hand side is $-P(1-\frac{P}{5})$, so the critical points are $P=0$ and $P=5$. The sign chart looks like:

P	0		5	
$-P\left(1-\frac{P}{5}\right)$		−		+
inc/dec		↙		↗

From this we see that the critical point $P=0$ is stable, and the critical point $P=5$ is unstable.

Our work agrees with the direction field



The biological interpretation of our findings is the following: Suppose that we have a population which we believe will be unable to reproduce – and therefore will die off – if there are fewer than 5 individuals. (This happens to populations that reproduce in large groups, or that live on large territories and might have trouble meeting a mate.) Then this population can be modeled by the threshold growth model with threshold level $m = 5$.

In mathematical words, we can say that if we have an initial value $P(t_0) = P_0 < 5$ then the population will approach 0 as $t \to \infty$. If the initial value is $P(t_0) = 5$, then the population remains at 5 forever. Finally, if the initial value is $P(t_0) = P_0 > 5$, then the population will increase without bounds as $t \to \infty$.

We note here that this is slightly different than the threshold growth model given in Section 7.5 of the book. We prefer this threshold growth model since ours has $P = 0$ as a critical point.

5.8 Things you need to know

You should become very comfortable with all of the vocabulary introduced this week (in bold in this text). In addition, you should remember Euler's method. Write down Euler's method below: