

The Hanging Chain Problem

Consider a chain consisting of n one unit length links. If we suppose the ends of the chain are supported at the same height and are a distance L apart then the problem may be formulated as:

$$\begin{array}{ll} \underset{y \in \mathbb{R}^n}{\text{minimize}} & c^T y \\ \text{s.t.} & e^T y = 0, \quad \sum_{i=1}^n \sqrt{1 - y_i^2} = L, \end{array}$$

where $c_i = n - i + \frac{1}{2}$.

You may if you wish consider more elaborate forms of the problem such as having the ends at different heights, having the links of different weights or even having some object that interferes with the chain hanging freely. What we are doing is minimizing the potential energy of the chain. The element y_i is the horizontal distance of the end of the i th link from that of the end of the $(i - 1)$ th link.

The above is one of many equivalent mathematical formulations of the hanging chain problem. The purpose of the project is in part a learning experience to illustrate that the choice of formulation impacts the behavior of algorithms. We shall learn in the unconstrained case that the condition number of the Hessian impacts the rate of convergence of the steepest descent algorithm and also the rate of convergence of Newton's (and hence modified Newton's) method. Likewise for linearly constrained problems the condition number of the reduced Hessian and the condition of the constraint matrix corresponding to the constraints active at the solution impacts the rate of convergence of methods for linearly constrained problems. There are also other factors. For example, we know that the ideal objective is a quadratic function since that is our model function. It follows the less like a quadratic that the objective looks like the less successful will be our methods. Size is another issue. Usually it is easier to solve smaller problems *all other things being equal*. Usually it better not to have nonlinear constraints, equality constraints are preferable to inequality constraints, etc. Sometimes the properties of a particular formulation may be deduced by considering limiting cases. In the hanging chain problem it is of interest to consider the nature of the chain as the number of links grows.

What you need to do is to try solving the problem first by using the above formulation. Comment on the reasons for the deterioration in the behavior of the algorithm as the number of links increases. Try another formulation and note the corresponding behavior of the algorithm. What would be nice is a formulation for which the number of iterations did not increase significantly as the number of links grows. That may be asking a lot but hopefully it does not grow more than at a linear rate, which is the rate we would get for a quadratic function. You may like to experiment with starting at a near solution as opposed to a poor initial estimate. Try and use your imagination.