Numerical Approximations of Partial Differential Equations in Theory and Practice

CME 325, winter 2008

CME325

Course consists of 12 lectures, 4 homework sets, a project.

Requirements: written solutions to problem sets + a written project report (project is for the 2 credit option).
Grades will be based on the quality of solutions and report

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Office hours: Mondays and Wednesdays 3-4 pm, room Durand 105b

Reading:
Gustafsson, *High Order Difference methods for Time Dependent PDE*, Springer, 2008
Or
Gustafsson, Kreiss, Oliger, *Time dependent problems and difference methods*, Wiley, 1995

Lectures, Mo-Wed-Fr 12:35pm-2:05pm in Hewlett103

1. Initial-value problems (IVP), well-posedness and stability,
2. Initial-boundary-value problems (IBVP), well-posedness and stability by energy estimtes
3. Stability, convergence and accuracy for IBVP
4. High order methods in space and time
5. Space-time coupled methods
6. Analysis of errors
7. Non-regular grids
8. Implementation of boundary conditions for parabolic IBVP
9. Implementation of boundary conditions for hyperbolic IBVP
10. Non-reflecting boundary conditions and normal-mode analysis
11. Discrete non-reflecting boundary conditions
12. Non-regular domains

Well-posedness for time-dependent pde

$$u_t = Pu + F(x,t), t \ge 0, x \in \Omega$$
$$u(x,0) = f(x)$$
$$Bu\big|_{x\in\partial\Omega} = g(x,t)$$

existence, uniqueness, computability (≈solution depends smoothly on data) ?

An energy estimate:

$$\|u(\cdot,t)\|^2 \le K(t)(\|f\|^2 + \int_0^t \|F(\cdot,\tau)\|^2 + \|g(\tau)\|^2_{\partial\Omega} d\tau)$$

- On bounded intervalls: small perturbations in data have small effect
- Uniqueness in L_2 sense

2π-Periodic IVP

$$u_t = Pu + F(x,t), t \geq 0, -\infty < x < \infty$$

$$u(x,0) = f(x)$$

$$u(x,t) = u(x+2\pi,t)$$

$$u_j^{n+1} = Qu_j^n + kF_j^n, j = 0,1,..N, n = 0,1,..$$

$$u_j^0 = f_j, \qquad u_j^n = u_{j+N}^n$$

$$u_j^n \approx u(x_j, t_n), \quad f_j = f(x_j)$$

$$x_j = jh, \quad h = 2\pi / N$$

$$t_n = nk,$$

$$\|u(\cdot,t)\|^2 = \int_0^{2\pi} |u(x,t)|^2 dx,$$

$$\|u^n\|_h^2 = \sum_0^N |u_j^n|^2 h$$

Well-posedness and stability of IVP

$$u_t = Pu + F(x,t),$$
$$u(x,0) = f(x)$$

Def : well - posed if for $F \equiv 0$

unique solution exists with

$$\|u(\cdot,t)\|^2 \leq Ke^{\alpha t}\|f\|$$

$$u_j^{n+1} = Qu_j^n + kF_j^n,$$
$$u_j^0 = f_j$$

Def : stable if for $F \equiv 0$

solution satisfies

$$\left\|u^n\right\|_h^2 \leq Ke^{\alpha t_n}\|f\|_h^2$$

F≠0 estimated by duhamel's principle

Fourier analysis for constant coefficients

$$u(x,t) = \sum_{-\infty}^{\infty} \hat{u}(\omega,t) e^{i\omega t} \text{ in IVP}$$

$$\hat{u}_t = \hat{P}(i\omega)\hat{u}, \hat{u}(\omega,0) = \hat{f}(\omega)$$

has solution

$$\hat{u}(\omega,t) = e^{\hat{P}(i\omega)t} \hat{f}(\omega)$$

Thm : IVP well-posed iff

$$\exists \mathrm{K}, \alpha : \left| e^{\hat{P}(i\omega)t} \right| \leq K e^{\alpha \mathrm{t}}$$

Petrovsky condition is necessary :

$$\mathrm{Re}(\lambda(\hat{P}(i\omega))) \leq \alpha$$

$$u_j^n = \sum_{-N/2}^{N/2} \hat{u}_\omega^n e^{i\omega x_j} \text{ in discrete IVP}$$

$$\hat{u}_\omega^{n+1} = \hat{Q}(\omega h)\hat{u}_\omega^n, \hat{u}_\omega^0 = \hat{f}_\omega$$

Thm : discrete IVP stable iff

$$\left| \hat{Q}(\xi) \right| \leq 1 + ak \ \forall \ |\xi| \leq \pi$$

von Neuman condition is necessary :

$$\left| \lambda(\hat{Q}(i\omega)) \right| \leq 1 + \tilde{a}k$$

Sufficient if uniformally diagonalizable

Semi-discrete IVP

$$\frac{du_j}{dt} = Qu_j + F_j\,, \qquad u_j = f_j$$

Def : semi - discrete IVP is stable if for $F \equiv 0$

$$\left\| u(t) \right\|_h^2 \le Ke^{\alpha t} \left\| f \right\|_h^2$$

$$\frac{d\hat{u}_\omega}{dt} = \hat{Q}(\omega h)\hat{u}_\omega, \qquad \hat{u}_\omega(0) = \hat{f}_\omega$$

von Neuman condition : $\left|\mathrm{Re}(\lambda(\hat{Q}(\xi))\right| \le \tilde{\alpha} \;\forall\; \left|\xi\right| \le \pi$

sufficient if $\hat{Q}(\omega h)$ diagonalizable

Standard IVP are well-posed

Hyperbolic:

$\mathbf{u_t+u_x=0}$,

$u_t+\alpha u_x+\beta u_y=0$

$u_t=Au_x$,

$u_t=Au_x+Bu_y$

Parabolic:

$\mathbf{u_t=u_{xx}}$,

$u_t=-u_{xxxx}$,

$u_t=u_{xx}+u_{yy}$

$\mathbf{u_t+u_x=u_{xx}}$

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Well-posedness for these is not altered by lower order terms

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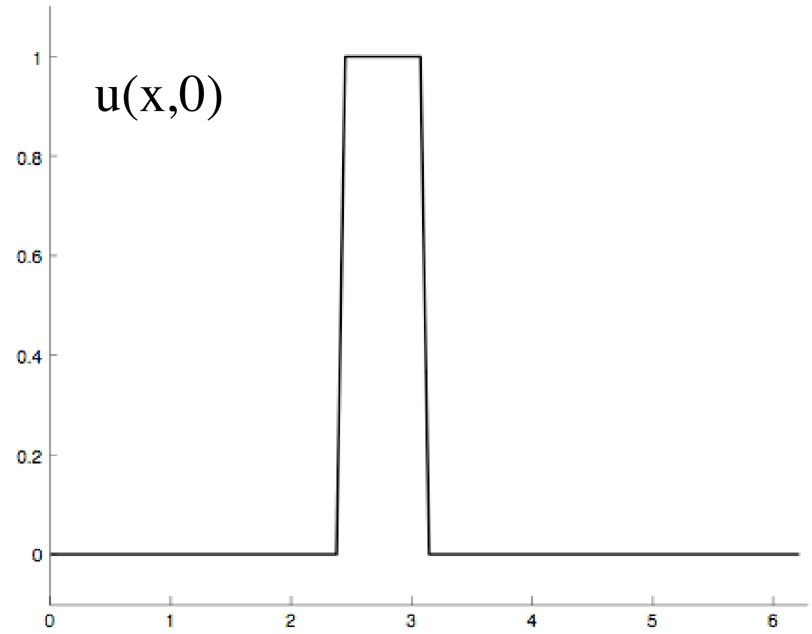

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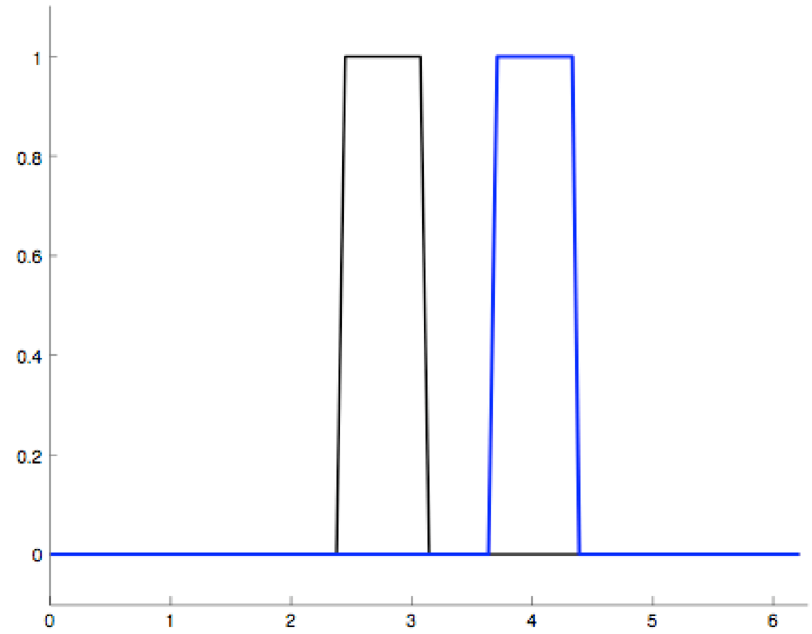

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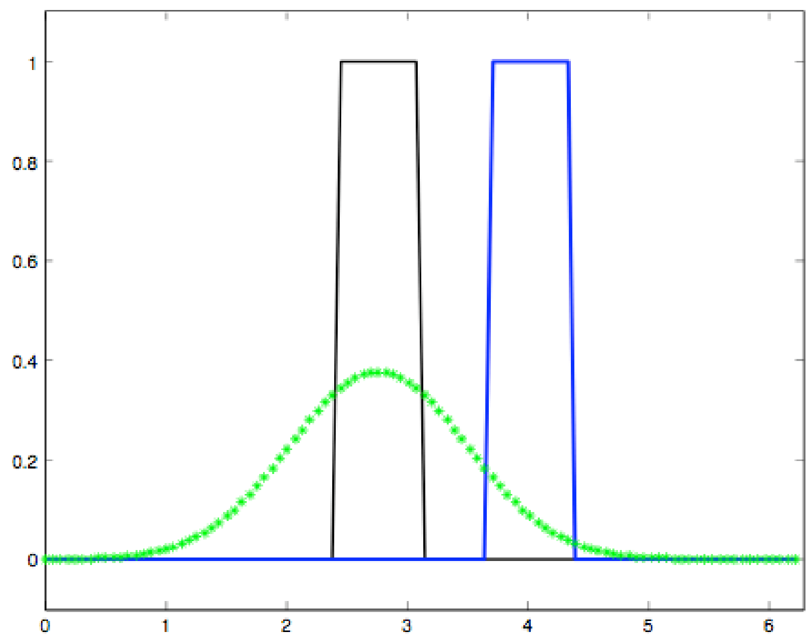

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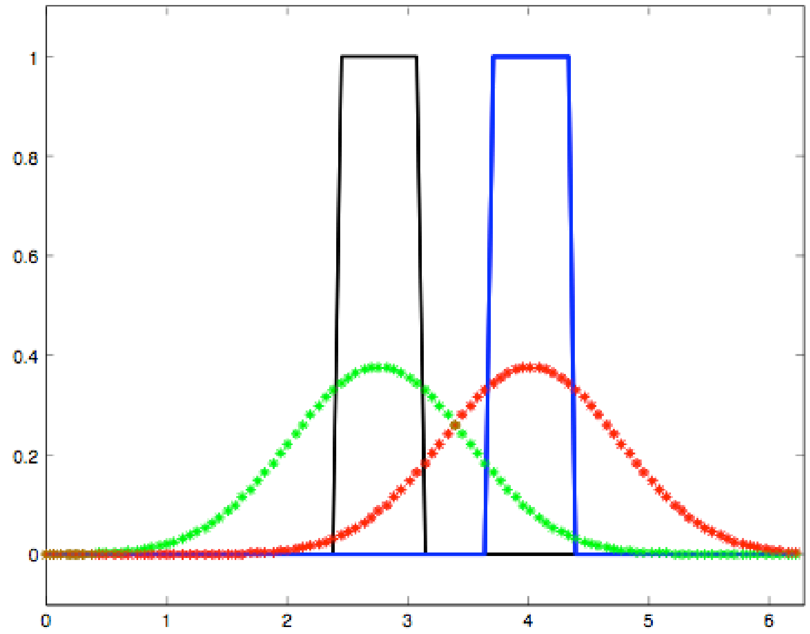

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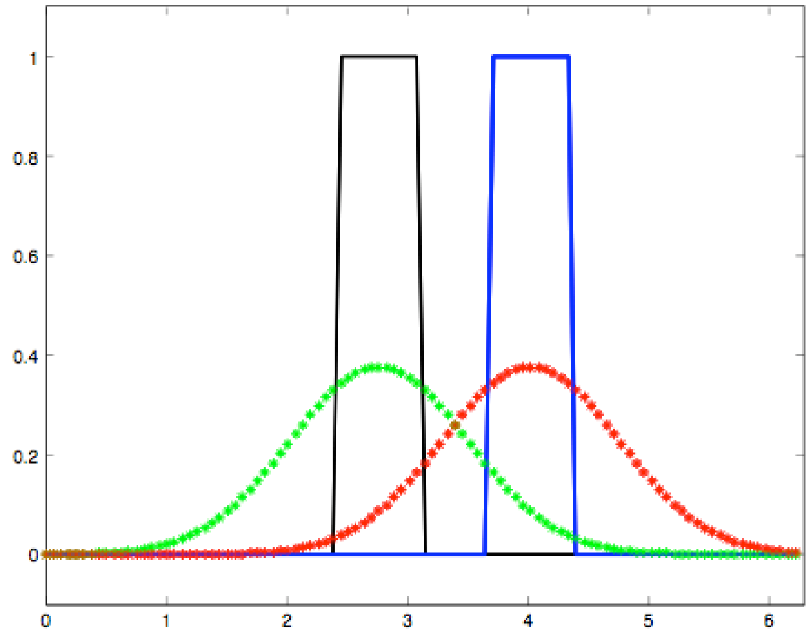

•Variable coefficients? try energy estimate directly!