

Numerical Approximations of Partial Differential Equations in Theory and Practice

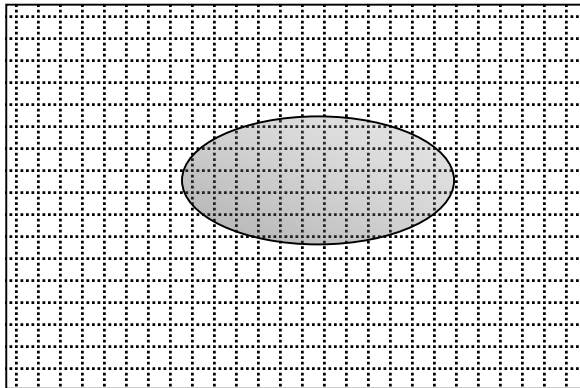
CME 325, winter 2008

Lecture 10

CME325, winter 08, Friday Feb 22

1. Initial-value problems (IVP), well-posedness and stability, periodic problems
2. Initial-boundary-value problems (IBVP), well-posedness and stability by energy estimates
3. Stability, Convergence and Accuracy
4. High order discretizations in time and in space
5. Coupled methods + how to measure accuracy
6. Errors: Damping, Dispersion, Conservation
7. Implementation of boundary conditions for hyperbolic IBVP
8. More applications of SBP+SAT
9. Boundary conditions for parabolic IBVP+normal mode analysis
- 10. Imbedded boundaries and interfaces**
11. Non-reflecting boundary conditions

Embedded boundaries



Grid is not aligned with boundary

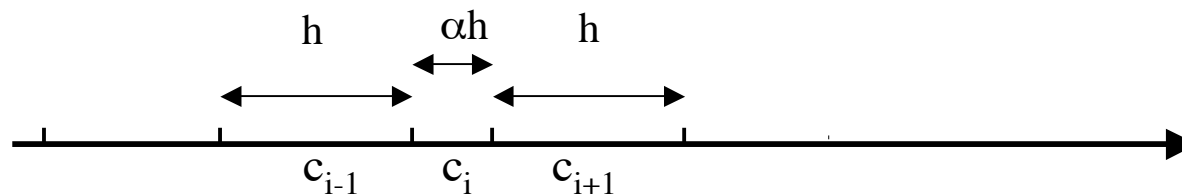
- + no need for special meshes
- implementing boundary conditions is more difficult
- accuracy and stability difficult to achieve and analyze

- Cut cells approach: may led to “bad cells” and severe stability restrictions
- Ghost points/cells approach: No general technique for stability and high order accuracy

H-box method in 1D

LeVeque,
Berger, Helzel

Cauchy problem: $u_t + u_x = 0$, $u(x, 0) = f(x)$



Fluxes are computed using information in equal size boxes.
Numerical domain of influence is extended at small grid cell.
Stability?

Normal mode analysis, fully discret case

$$u_j^{n+1} = Qu_j^n, j = 1, \dots$$

$$B_h u^n = 0, \quad \|u^n\|_h < \infty$$

Look for solutions $\phi_j z^n$ where z is a complex number:

$$z\phi_j = Q\phi_j, j = 1, \dots$$

$$B_h \phi = 0$$

$$\|\phi\|_h < \infty$$

Eigenvalue problem for z and gridfunction ϕ

If there is an eigenvalue z_0 with $|z_0| > 1$ the discretization is unstable. (Why?)

Normal mode analysis,

fully discret case

$$z\phi_j = hQ\phi_j, j=1, \dots$$

$$B_h\phi_0 = 0, \|\phi\|_h < \infty$$

Determinant condition:

- Assume $|z| > 1$
- Determine general solution of difference equation satisfying $\|\phi\|_h < \infty$
- Apply B_h yields $C(z)\sigma = 0$
- ***Det C(s) ≠ 0?***

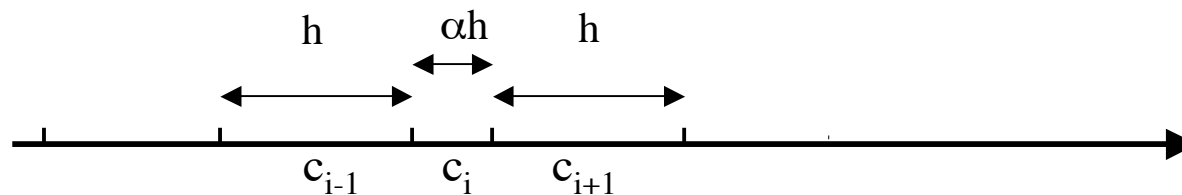
Godunov - Ryabenkii condition is necessary for stability :

$\text{Det } C(z) \neq 0$ for all $|z| > 1$

H-box method in 1D

LeVeque,
Berger, Helzel

Cauchy problem: $u_t + u_x = 0$, $u(x, 0) = f(x)$



Fluxes are computed using information in equal size boxes.
Numerical domain of influence is extended at small grid cell.
Stability is OK for Lax-Wendroff

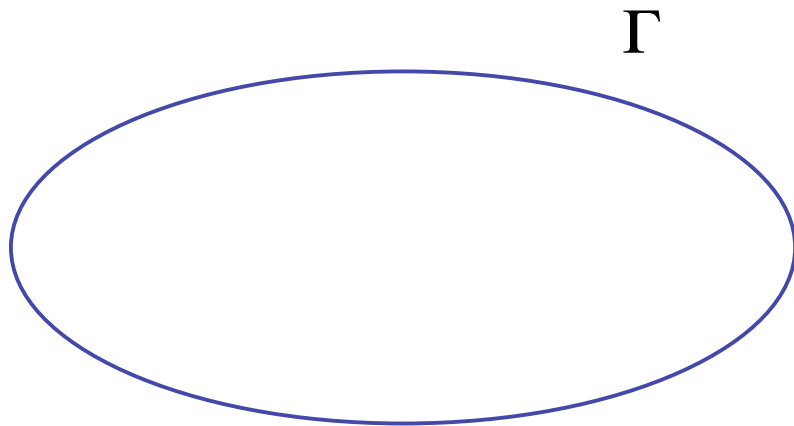
Extendable to small cells at boundaries

Incompressible Navier-Stokes equation

$$\nabla \cdot \mathbf{u} = 0,$$

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \mu \nabla^2 \mathbf{u} + F$$

x



Γ

- Elastic membrane
- Interface between fluids

$$F = f(x) \delta(\Gamma)$$

Alternative 1

- Use regularized delta function δ_h

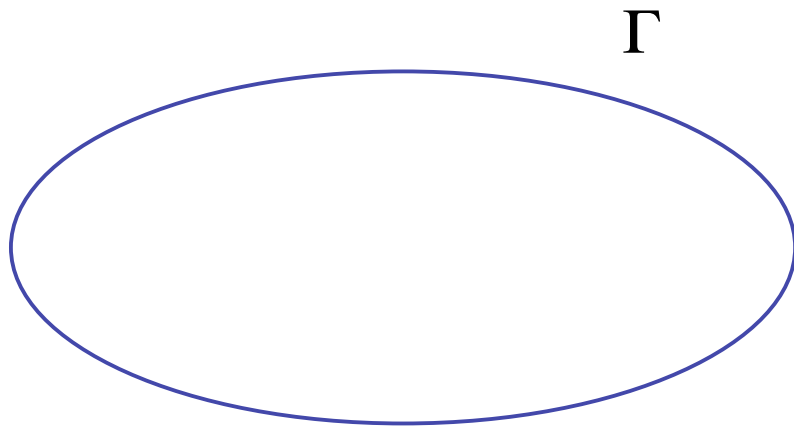
Immersed boundary method by Peskin, developed for simulating blood flow in the heart taking elastic walls into account, (Acta Numerica (2002) pp1-39) uses regularized delta function δ_h

Incompressible Navier-Stokes equation

$$\nabla \cdot \mathbf{u} = 0,$$

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \mu \nabla^2 \mathbf{u} + F$$

x



Γ

- Elastic membrane
- Interface between fluids

$$F = f(x) \delta(\Gamma)$$

Alternative 2

- Incorporate jumps explicitly

Model problem

$$p_{xx} = f(x) + c\delta'(x - \alpha),$$
$$p(0) = a, p(1) = b$$

x

Some References

- Berger, Helzel, LeVeque, *h-Box method for the approximation of hyperbolic conservation laws on irregular grids*, SIAM J Num Anal, Vol 41, No 3, 2003
- Peskin, *The immersed boundary method*, Acta Num 2002
- Mittal, Iaccarino, *Immersed Boundary Methods*, Annu. Rev. Fluid Mech, 2005
- Tornberg, Engquist, *Numerical approximations of singular source terms in differential equations*, JCP, Vol 200, 2004
- Lee, LeVeque, *An Immersed interface method for incompressible Navier-Stokes equations*, SIAM J Sc Comp, 2003
- H-O Kreiss, Petersson, *A second order accurate embedded boundary method for the wave equation with Dirichlet Data*, SIAM J Sc Comp, 2006
- Tseng, Ferziger, *A ghost cell immersed boundary method for flow in complex geometry*, JCP 192, 2003