

CME325, winter 08, Friday Feb 1

1. Initial-value problems (IVP), well-posedness and stability, periodic problems

$$u_j^{n+1} = u_j^n + kD_+D_-(I - \frac{h^2}{12}D_+D_-)u_j^n + kF_j^n,$$

$$u_j^0 = f_j, \quad u_{j+N}^n = u_j^n$$

2. **Initial-boundary-value problems (IBVP), well-posedness and stability by energy estimates**

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1. Initial-value problems (IVP), well-posedness and stability, periodic problems

$$u_j^{n+1} = Qu_j^n + kF_j^n,$$

$$u_j^0 = f_j, \quad u_{j+N}^n = u_j^n$$

Def : stable if for $F \equiv 0$ solution satisfies

$$\|u^n\|_h^2 \leq Ke^{\alpha t} \|f\|_h^2 \quad \forall f, K \text{ and } \alpha \text{ independent of } t, f$$

2. **Initial-boundary-value problems (IBVP), well-posedness and stability by energy estimates**

Well-posedness for IBVP

$$u_t = Pu + F(x, t), t \geq 0, 0 \leq x \leq 1$$

$$u(x, 0) = f(x)$$

$$B_0 u|_{x=0} = g_0(t), B_1 u|_{x=1} = g_1(t)$$

What boundary conditions?

Want existence, uniqueness, estimate

Def 1: wellposed if for $F \equiv 0$, $g \equiv 0$,

unique solution exists with

$$\|u(\cdot, t)\|^2 \leq Ke^{\alpha t} \|f\|$$

Well-posedness implies some estimate for $g \neq 0, F \neq 0$

”Good” boundary conditions

1D: $0 \leq x \leq 1$

$$u_t + u_x = F, \quad u(0, t) = g(t)$$

$$u_t = u_{xx}, \quad u(0, t) = g_0(t) \text{ \& } u_x(1, t) = g_1(t) \text{ or } \dots$$

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x, \quad \text{prescribe ingoing characteristic variables}$$

2D: $0 \leq x \leq 1,$

periodic in y

Use 1-D theory

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_y, \quad \text{same as above}$$

$$u_t = u_{xx} + u_{yy}, \quad \text{same as above}$$

There are other ”good” boundary conditions!

Semi-discrete IBVP

$$\frac{du_j}{dt} = Qu_j + F_j, j = 1, \dots, N-1$$

$$B_h u = g(t)$$

$$u_j(0) = f_j, j = 1, \dots, N-1$$

Def : semi - discrete IBVP is stable if for $F \equiv 0, g \equiv 0$

$$\|u(t)\|_h^2 \leq Ke^{\alpha t} \|f\|_h^2$$

$F \neq 0$ estimated by duhamel's principle
 $g \neq 0$?

Summation by parts

$$\sum_1^{N-1} \langle u, D_+ v \rangle = - \sum_2^N \langle D_- u, v \rangle + u_N v_N - u_1 v_1$$

$$\sum_1^{N-1} \langle u, D_0 v \rangle = - \sum_2^N \langle D_0 u, v \rangle + \frac{1}{2}(u_{N-1} v_N + u_N v_{N-1}) - \frac{1}{2}(u_0 v_1 + u_1 v_0)$$

Check this!

Semi-discrete IBVP

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$$B_h u = g(t)$$

$$u_j(0) = f_j, j = 1, \dots, N-1$$

Def : let \mathcal{V}_h be the space of grid functions satisfying $B_h v = 0$.

Then Q is semi-bounded on \mathcal{V}_h if for all v in \mathcal{V}_h if

$$\operatorname{Re} (v, Qv) \leq \alpha (v, v)$$

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Thm: If Q is semi-bounded for some inner product

$$(u, v)_h = h \sum_1^{N-1} q_j \langle u_j, H v_j \rangle, \quad q_j > 0, \quad H = H^T > 0$$

Then the semi-discrete IBVP is stable

Well-posedness for IBVP

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$$u(x, 0) = f(x)$$

$$B_0 u|_{x=0} = g_0(t), B_1 u|_{x=1} = g_1(t)$$

Def 2: let \mathcal{V} be the space of differentiable functions satisfying the boundary conditions. Then P is semi-bounded on \mathcal{V} if for all v in \mathcal{V} if $Re (v, Pv) \leq \alpha (v, v)$

Def 3: P is maximally semi-bounded on \mathcal{V} if semi-bounded but not semi-bounded with fewer boundary conditions

Discrete product rule

$$hD_-(a_j b_j) = a_j D_- b_j + (D_- a_j) E^{-1} b_j$$

$$hD_+(a_j b_j) = (E a_j) D_+ b_j + (D_+ a_j) b_j$$

$$hD_0(a_j b_j) = a_j D_0 b_j + \frac{1}{2} \left((D_+ a_j) E b_j + (D_- a_j) E^{-1} b_j \right)$$

Check this!

More useful tricks

$$|a \cdot b| \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2, \text{ any } \varepsilon > 0$$

$$\int a(x)b(x)dx \leq \varepsilon \int a^2 dx + \frac{1}{4\varepsilon} \int b^2 dx$$

$$\sum a_j b_j \leq \varepsilon \sum a_j^2 + \frac{1}{4\varepsilon} \sum b_j^2$$

$$|D_+ a_j| \leq |a_x|_\infty$$

Stability concepts

- Wellposed IVP
- Stable discretization of IVP
- Stable semi-discretization of IVP
- Wellposed IBVP
- Stable discretization of IBVP
- Stable semi-discretization of IBVP
- Strictly stable discretizations

- Time stable IVP
- Time stable IBVP

$$\lim_{t \rightarrow \infty} \|u(\cdot, t)\| \leq K$$