CME325, winter 08, Monday Feb 4

1. Initial-value problems (IVP), well-posedness and stability, periodic problems

2. Initial-boundary-value problems (IBVP), well-posedness and stability by energy estimates

3. Stability, Convergence and Accuracy

2π-Periodic IVP

$$u_t = Pu + F(x,t), \quad t \ge 0,$$
$$u(x,0) = f(x)$$
$$u(x,t) = u(x+2\pi, t)$$

$$u_j^{n+1} = Qu_j^n + kF_j^n, \quad j = 0,1,..N$$
$$u_j^0 = f_j,$$
$$u_j^n = u_{j+N}^n$$

$$u(x_j, t_n) - u_j^n \ ?$$

Assume well-posedness and stability

$$\left\| u^n \right\|_h^2 \le Ke^{\alpha t_n} \left(\left\| f \right\|_h^2 + \sum_{\nu=0}^{n-1} k \left\| F^\nu \right\|_h^2 \right)$$

Truncation error

Apply the discrete formula to the true, smooth solution

$$u(x_j, t_{n+1}) = Qu(x_j, t_n) + kF(x_j, t_n) + k\tau(x_j, t_n)$$

$$u(x_j, 0) = f(x_j) + \phi_j,$$

τ is called **truncation error**

If $|\tau(x_j,t_n)| \leq C(h^p+k^q)$ the **order of accuracy** is *(p,q)*

(use Taylor expansion)

The approximation is **consistent** if $p>0$, $q>0$

Theorem

Assume the PDE is well-posed with smooth solution *u(x,t),* the difference approximation is stable and consistent with accurate of order *(p,q),* and that the truncation error in initial data is also of order *(p,q)*. Then for any *T* the error satisfies

$$\left\| u(x,t_n) - u^n \right\|_h = K(T)(h^p + k^q), \quad 0 \le t \le T$$

Here *C(T)* depends only on *T* and on the smooth *u*.

Can have convergence even without smooth solution
Can allow less accuracy in initial data

Lax Equivalence Theorem

If the PDE is well-posed and the difference approximation is consistent, then the approximation is convergent if and only if the approximation is stable.

IBVP

$$u_t = Pu + F(x,t), \quad t \ge 0, \quad 0 \le x \le 1,$$
$$u(x,0) = f(x)$$
$$B_0 u(0,t) = g_0(t)$$
$$B_1 u(1,t) = g_1(t)$$

$$u_j^{n+1} = Qu_j^n + kF_j^n, \quad n = 0,1,..$$
$$u_j^0 = f_j,$$
$$\tilde{B}_0 u_0^n = g_0(t),$$
$$\tilde{B}_N u_N^n = g_1(t)$$

$$u(x_j, t_n) - u_j^n \ ?$$

Must have estimate for inhomogeneaus boundary condition!

Strong stability for IBVP

$$u_j^{n+1} = Qu_j^n + kF_j^n,$$

$$\mathrm{B_h u = g(t)}$$

$$u_j^0(0) = f_j,$$

Def : The discrete IBVP is strongly stable if

$$\left\|u^n\right\|_h^2 \quad \le Ke^{\alpha t_n}(\|f\|_h^2 + \sum_{\nu=0}^{n-1} k(\left\|F^\nu\right\|_h^2 + \left|g^\nu\right|^2)$$

Hyperbolic problems: derive estimate by energy method

If not strongly stable?

Reformulate to get homogeneaus boundary condition:

Find

$$\phi_j^n : \ \tilde{B}_0 \phi_0^n = g_0^n, \ \ \tilde{B}_1 \phi_1^n = g_1^n$$

then

$$v_j^n = u_j^n - \phi_j^n$$

Satisfies homogeneaus boundary conditions
with extra forcing in equation

$$\left\| v^n \right\|_h^2 \le C(t_n) \left(\left\| f \right\|_h^2 + \sum_0^{n-1} k \left(\left| g_0(t_\nu) \right|^2 + \left| g_1(t_\nu) \right|^2 + \left| g_0{}'(t_\nu) \right|^2 + \left| g_1{}'(t_\nu) \right|^2 + \left\| F^\nu \right\|_h^2 \right) \right)$$

If stable discretization of well-posed IBVP then

$$\left\| u(x,t_n) - u^n \right\|_h = K(T)(h^{p^*} + k^q), \quad 0 \le t \le T$$

$$p^* = min(p_{interior}, p_{boundary})$$

Can allow lower order in Q near boundary!