

CME325, winter 08, Wednesday Feb 6

1. Initial-value problems (IVP), well-posedness and stability, periodic problems
2. Initial-boundary-value problems (IBVP), well-posedness and stability by energy estimates
3. Stability, Convergence and Accuracy
4. **High order discretizations in time and in space**

Method of Lines

$$u_t = Pu + F(x, t), t \geq 0,$$

$$u(x, 0) = f(x)$$

$$u(x, t) = u(x + 2\pi, t)$$

1. Discretize in space: $u_t = Qu + F$
(semidiscretization)
2. Use ODE method in time

High order in space by using many points

Ex:

Differential operator	Central difference operator	order
$\frac{\partial}{\partial x}$	D_0	2
	$D_0(I - \frac{h^2}{6} D_+ D_-)$	4
	$D_0(I - \frac{h^2}{6} D_+ D_- + \frac{h^2}{30} D_+^2 D_-^2)$	6
$\frac{\partial^2}{\partial x^2}$	$D_+ D_-$	2
	$D_+ D_- (I - \frac{h^2}{12} D_+ D_-)$	4
	$D_+ D_- (I - \frac{h^2}{12} D_+ D_- + \frac{h^2}{90} D_+^2 D_-^2)$	6

Non-smooth solutions: may lose accuracy, "wiggles"

Boundaries need special treatment (More later!)

Multi-dimension: straight forward

Stability

use Fourier analysis or check semi-boundedness

$$\text{ex : } u_t = \frac{1}{2} (a(x)u_x + (a(x)u)_x)$$

$$\frac{du_j}{dt} = \frac{1}{2} a_j D_0 \left(I - \frac{h^2}{6} D_+ D_- \right) u_j + \frac{1}{2} D_0 \left(I - \frac{h^2}{6} D_+ D_- \right) a_j u_j$$

Use summation by parts!

Theorem: If Q is semi-bounded on the space of periodic grid functions then $u_t = Qu + F$ is a stable semi-discretization. (prove this!)

High order in space by using Pade' operators

Pade' order 4: $u_t = u_x$ is semi-discretized by

$$u_t = Qu,$$

where

$$Qu = (I + (h^2/6) D_+ D_-)^{-1} D_0 u$$

To use: introduce gridfunction $u_x = Qu$

- Compute right hand side: $D_0 u$
- Solve tridiagonal, symmetric system for u_x :

$$(I + (h^2/6) D_+ D_-) u_x = D_0 u$$

+ compact,

+ truncation error has small coefficient

- must solve system

Discretize in time

Explicit:

Runge-Kutta methods

Adams-Bashforth methods

Leap-Frog

$$u^{n+1} = \left(I + kQ + \frac{k^2}{2}Q^2 + \frac{k^3}{6}Q^3 + \frac{k^4}{24}Q^4 \right) u^n$$

$$u^{n+1} = u^n + kQ \left(\frac{3}{2}u^n - 2u^{n-1} + \frac{1}{2}u^{n-2} \right)$$

Implicit:

BDF

(Backward Differentiation Formula)

Crank-Nicholson

$$\left(\frac{3}{2}I - kQ \right) u^{n+1} = 2u^n - \frac{1}{2}u^{n-1}$$

$$\left(I - \frac{1}{2}kQ \right) u^{n+1} = \left(I + \frac{1}{2}kQ \right) u^n$$

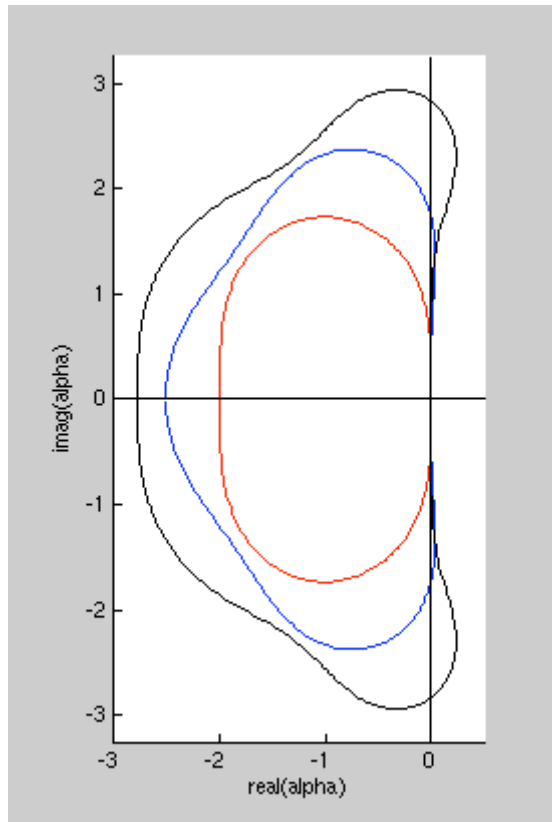
How choose a method and a time step?

Use Fourier analysis !

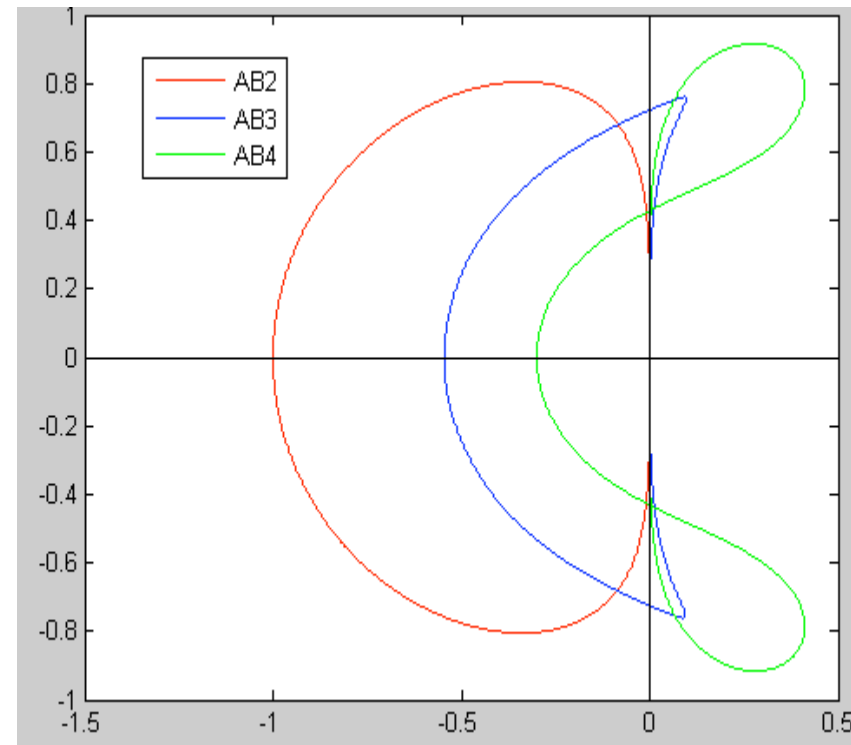
Stability regions for explicit methods

Runge-Kutta

RK4 RK3 RK2



Adams-Bashforth



Part of the imaginary axis is included!

Rule of thumb:

Based on analysis for periodic, constant coefficients

Hyperbolic problems Use explicit method with part of imaginary axis in stability region:

Runge-Kutta 3, 4 or Adams-Bashforth 2,3,4

Parabolic problems Use implicit method with negative real axis in stability region:

BDF 2-6 or Crank-Nicholson

IBVP with variable coefficients

Theorem for hyperbolic IBVP:

If semi-discretization is semi-bounded then RK3, RK4, AB2-4 are stable for $\|kQ\|_h \leq R^*$
(R^* is related to stability region)

In practice: try time step determined by Fourier analysis of frozen coefficient problems.

Boundary treatment may restrict further!