Numerical Approximations of Partial Differential Equations in Theory and Practice

CME 325, winter 2008

Lecture 5

CME325, winter 08, Friday Feb 8

1. Initial-value problems (IVP), well-posedness and stability, periodic problems
2. Initial-boundary-value problems (IBVP), well-posedness and stability by energy estimates
3. Stability, Convergence and Accuracy
4. High order discretizations in time and in space
5. **Coupled methods + how to measure accuracy**

Coupled methods

Some methods are not based on method of lines.

Example of discretizations of $u_t = u_x$.

$$\text{Lax - Wendroff}: \quad u_j^{n+1} = u_j^n + \tfrac{k}{h} D_0 u_j^n + \frac{k^2}{2h^2} D_+ D_- u_j^n,$$

$$\text{Lax - Friedrichs}: \quad u_j^{n+1} = \tfrac{1}{2}(u_{j+1}^n + u_{j-1}^n) + \tfrac{k}{h} D_0 u_j^n$$

Lax-Wendroff is based on Taylor series expansion, and can be generalized to higher order and to non-linear systems: MacCormack scheme for the Euler equations

How measure order of convergence?

1. Analyze truncation error
2. Compare approximate solutions with a known solution *U*(x,t)*. Note: given $u_t = Pu$ you can construct a problem with solution *U*(x,t)* (Choose!)
3. Compare approximations with each other.

Determine p and q in :

$$u_j^n = u(x_j, t_n) + a(x_j, t_n)h^p + b(x_j, t_n)k^q + O(h^{p+1} + k^{q+1})$$

When stability restriction $k \le \lambda h$ fix $k = ch$:

$$u_j^n = u(x_j, t_n) + \tilde{a}(x_j, t_n)h^{\tilde{p}} + O(h^{\tilde{p}+1})$$

How measure order of convergence?

Great care must be taken to compare at the same point in time and space!

	Approx of $u^*(x)=u(x,t^*)$	Pointwise estimate of order with known solution	Pointwise estimate, no known solution
$h_0,\ k_0=\lambda h_0$ $h_0/2,\ k_0/2$ $h_0/4,\ k_0/4$	$u^{(0)}$ $u^{(1)}$ $u^{(2)}$	$\tilde{p}_1 = \log_2 \frac{u_j^{(0)} - u^*(\bar{x})}{u_{j'}^{(1)} - u^*(\bar{x})}$ $\tilde{p}_2 = \log_2 \frac{u_{j'}^{(1)} - u^*(\bar{x})}{u_{j''}^{(2)} - u^*(\bar{x})}$	$\tilde{p} = \log_2 \frac{u_j^{(0)} - u_{j'}^{(1)}}{u_{j'}^{(1)} - u_{j''}^{(2)}}$
$h_0,\ k_0$ $h_0/2,\ k_0$ $h_0/4,\ k_0$	$u^{(0)}$ $u^{(1)}$ $u^{(2)}$		$p = \log_2 \frac{u_j^{(0)} - u_{j'}^{(1)}}{u_{j'}^{(1)} - u_{j''}^{(2)}}$
$h_0,\ k_0$ $h_0,\ k_0/2$ $h_0,\ k_0/4$	$u^{(0)}$ $u^{(1)}$ $u^{(2)}$		$q = \log_2 \frac{u_j^{(0)} - u_{j'}^{(1)}}{u_{j'}^{(1)} - u_{j''}^{(2)}}$

p's and *q*'s should converge with further refinement!

Use norms instead of pointwise estimates

Great care must be taken to compare at the same point in time and space!

	Approx of $u^*(x)=u(x,t^*)$	Pointwise estimate of order with known solution	Pointwise estimate, no known solution
h_0, $k_0=\lambda h_0$ $h_0/2$, $k_0/2$ $h_0/4$, $k_0/4$	$u^{(0)}$ $u^{(1)}$ $u^{(2)}$	$\tilde{p}_1 = {}_2\log \frac{\lVert u^{(0)} - u^* \rVert_{h_0}}{\lVert u^{(1)} - u^* \rVert_{h'}}$ $\tilde{p}_2 = {}_2\log \frac{\lVert u^{(1)} - u^* \rVert_{h'}}{\lVert u^{(2)} - u^* \rVert_{h''}}$	$\tilde{p} = {}_2\log \frac{\lVert u^{(0)} - u^{(1)} \rVert_{h'}}{\lVert u^{(1)} - u^{(2)} \rVert_{h''}}$
h_0, k_0 $h_0/2$, k_0 $h_0/4$, k_0	$u^{(0)}$ $u^{(1)}$ $u^{(2)}$		$p = {}_2\log \frac{\lVert u^{(0)} - u^{(1)} \rVert_{h'}}{\lVert u^{(1)} - u^{(2)} \rVert_{h''}}$
h_0, k_0 h_0, $k_0/2$ h_0, $k_0/4$	$u^{(0)}$ $u^{(1)}$ $u^{(2)}$		$q = {}_2\log \frac{\lVert u^{(0)} - u^{(1)} \rVert_{h'}}{\lVert u^{(1)} - u^{(2)} \rVert_{h''}}$

More about Errors

- Damping errors
- Dispersion errors
- Conservation errors

Techniques:

- Discrete Fourier analysis
- Modified equation