CME325, winter 08, Monday Feb 11

1. Initial-value problems (IVP), well-posedness and stability, periodic problems
2. Initial-boundary-value problems (IBVP), well-posedness and stability by energy estimates
3. Stability, Convergence and Accuracy
4. High order discretizations in time and in space
5. Coupled methods + how to measure accuracy
6. **Errors: Damping, Dispersion, Conservation**

Errors

Scalar, constant coefficients:

$$u_t = Pu,$$

$$\hat{u}(t+k) = e^{\hat{P}(i\omega)k}\hat{u}(t),$$

$$e^{\hat{P}(i\omega)k} = e^{\mathrm{Re}(\hat{P}(i\omega)k)}e^{i\,\mathrm{Im}(\hat{P}(i\omega)k)}$$

$$u_j^{n+1} = Qu_j^n,$$

$$\hat{u}^{n+1} = \hat{Q}(\xi)\hat{u}^n, \xi = \omega h \in [-\pi, \pi]$$

$$\hat{Q}(\xi) = \left|\hat{Q}(\xi)\right|e^{-i\alpha(\xi)k}$$

Damping error: $\left|\hat{Q}(\xi)\right| = e^{\mathrm{Re}(\hat{P}(i\omega)k)}$?

Dispersion error: $\alpha(\xi)k = \mathrm{Im}(\hat{P}(i\omega))$?

Small errors for small $|\xi|$ by consistency

Definition: A method is **dissipative** if (all eigenvalues of)

$$|Q(\xi)| \le 1 - \delta|\xi|^{2r} \qquad \delta > 0,\ |\xi| \le \pi,\ r > 0$$

diagonalizable systems ok, variable coefficients by freezing coefficiens

Leap frog

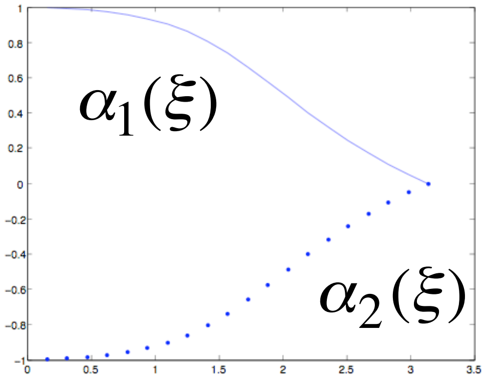
$$u_j^{n+1} = u_j^{n-1} - 2kD_0 u_j^n$$

Eigenvalues of the symbol:

$$q_1(\xi) = -i\tfrac{k}{h}\sin(\xi) + \sqrt{1-(\tfrac{k}{h}\sin(\xi))^2} = e^{-i\omega k\alpha_1(\xi)}$$

$$q_2(\xi) = -i\tfrac{k}{h}\sin(\xi) - \sqrt{1-(\tfrac{k}{h}\sin(\xi))^2} = e^{-i\omega k\alpha_2(\xi)}$$

Non-dissipative



k=0.8h

Modified equation

Assume v(x,t) satisfies discretiztion exactly
What PDE does v satisfy?

Finite Difference Methods for Ordinary and Partial Differential Equations, R.J. LeVeque. SIAM, 2007.
Numerical Methods for Conservation Laws, R. J. LeVeque, Lectures in Mathematics, ETH-ZurichBirkhauser-Verlag, 1990.

Conservative discretization

Conservation law: $u_t + f(u,u_x)_x = 0$ $\qquad u_j^{n+1} = Qu_j^n,$

Integral form: $\frac{d}{dt}\int_a^b u\,dx = -[f]_a^b$ $\qquad \sum_{j_a}^{j_b} h\left(u_j^{n+1} - u_{j+1}^n\right) = -k(f_a - f_b)$

only fluxes at boundary!

Example: $u_t + (c(x)u)_x = 0$

$D_-(cu)$ or $cD_-u + (D_0 c)u$?

Conservation essential for non-smooth solutions!

Non-conservative discretization for smooth solutions give small errors (by consistency)

Non-conservative discretization for non-smooth solutions give LARGE errors

Finite Volume methods

Conservation law: $u_t + f(u,u_x)_x = 0$ or $\frac{d}{dt}\int_a^b u\,dx = -[f]_a^b$

$$u_j^n \approx \frac{1}{h_j}\int_{I_j} u(x,t_n)dx$$

$$u_j^{n+1} = u_j^n - F_{j+1/2}^n + F_{j-1/2}^n,$$

+ always conservative discretization
+ can use non-uniform or unstructured grid
- in practice never higher order than 2

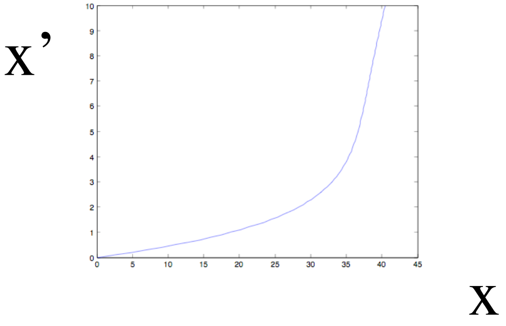
Non-uniform grids

Non-uniform gridpoints $x_1, x_2, \ldots x_N$

Finite Difference methods:

- Find transformation $x' = f(x)$: $x_i' = f(x_i)$ are uniform
- Change independent variable to x', yields variable coefficients
- Discretize on uniform grid

Ex $x = (10x' - 30)\, e^{0.5x'} + x' + 30$



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If transformation is non-smooth: may loose accuracy

Finite Volume method:

+ non-uniform grid easy,

- loose accuracy if grid is very non-uniform

Accuracy near boundary

$$\frac{du_j}{dt} = -D_0 u_j + F_j,\ j = 1, \ldots N-1 \qquad \tau_j = \mathcal{O}(h^2)$$

$$\frac{du_N}{dt} = -D_- u_N + F_N \qquad \tau_N = \mathcal{O}(h^1)$$

$$u_0 = g$$

Accuracy near boundary

$$\frac{dw_j}{dt} = -D_0 w_j + \tau_j,\ j = 1,\ldots N-1 \qquad \tau_j = \mathcal{O}(h^2)$$

$$\frac{dw_N}{dt} = -D_- w_N + \tau_N \qquad \tau_N = \mathcal{O}(h^1)$$

$$w_0 = 0$$

$$\frac{d}{dt}\|w\|_h^2 = 2(w,Qw)_h + 2(w,\tau)_h = -w_N^2 + hw_N\tau_N + 2\sum_1^{N-1} hw_j\tau_j$$

$$\frac{d}{dt}\|w\|_h^2 \le \tfrac{h^2}{4}\tau_N^2 + \sum h\tau_j^2 + \sum hw_j^2, \qquad \|w\|_h^2 \le O(h^4)$$

One order lower near boundary is ok!