CME325, winter 08, Wednesday Feb 13

1. Initial-value problems (IVP), well-posedness and stability, periodic problems
2. Initial-boundary-value problems (IBVP), well-posedness and stability by energy estimates
3. Stability, Convergence and Accuracy
4. High order discretizations in time and in space
5. Coupled methods + how to measure accuracy
6. Errors: Damping, Dispersion, Conservation
7. **Implementation of boundary conditions for hyperbolic IBVP**

high order finite difference methods for hyperbolic IBVP?

If solution smooth: Easy to get high order in interior by using wide stencil

At the boundary:

1. Reduce order locally, by more than one reduces overall accuracy
2. One-sided operators are in general unstable.

Systematic construction of high order finite difference methods for hyperbolic IBVP

1. high order wide stencil in interior
2. Special high order squew stencils near boundary
3. Satisfy the boundary conditions in a special procedure

Ex: 1 and 2 for $u_t + u_x = F$

$$\frac{du_0}{dt} = -D_+ u_0 + F_0$$

$$\frac{du_j}{dt} = -D_0 u_j + F_j, \; j = 1, \ldots N-1$$

$$\frac{du_N}{dt} = -D_- u_N + F_N$$

high order finite difference methods for hyperbolic IBVP

Definition: D is a first order **Summation-By-Parts** (SBP) operator if $Du \approx u_x$ and

$$2(u,Du)_{h'} = |u_N|^2 - |u_0|^2$$

for some inner product $(u,v)_{h'} = (u,Hv)_h$ where $H^T = H > 0$.

Here $(u,v)_h = \Sigma h u_i v_i$ is the standard discrete inner product.

SBP operator

$$hD_2^{(1)} = \begin{bmatrix} -1 & 1 & & & \\ -\frac{1}{2} & 0 & \frac{1}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & -\frac{1}{2} & 0 & \frac{1}{2} \\ & & & -1 & 1 \end{bmatrix} \qquad H = \begin{bmatrix} \frac{1}{2} & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \frac{1}{2} \end{bmatrix}$$

$$u_t + u_x = 0,\ 0 \le x \le 1,\ t \ge 0,$$

$$u(x,0) = f(t)$$

$$u(0,t) = g(t)$$

Let $\mathbf{u}$ be a grid function

$$\frac{d\mathbf{u}}{dt} = -D_2^{(1)}\mathbf{u}, \quad \mathbf{u}(0) = \mathbf{f}$$

Boundary condition?

Add penalty term!

SBP + SAT

simultaneaus approximation term

$$u_t + u_x = 0,\ 0 \le x \le 1,\ t \ge 0,$$
$$u(x,0) = f(t)$$
$$u(0,t) = g(t)$$

$$\frac{d\mathbf{u}}{dt} = -D^{(1)}\mathbf{u} - \boldsymbol{\tau}(u_N - \mathrm{g(t)})\mathrm{H}^{-1}\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}$$

$$\mathbf{u}(0) = \mathbf{f}$$

Accuracy? Will get formal order* if stable

Stability? Need $\tau \ge 0.5$

Large $\boldsymbol{\tau}$ will reduce allowable time step

* $p=min(p_{interior}, p_{boundary\ points} +1)$

SBP operators

$$hD_4^{(1)} = \begin{bmatrix} -\frac{24}{17} & \frac{59}{34} & -\frac{4}{17} & -\frac{3}{34} & & & & & \\ -\frac{1}{2} & 0 & \frac{1}{2} & & & & & & \\ \frac{4}{43} & -\frac{59}{86} & 0 & \frac{59}{86} & \frac{4}{43} & & & & \\ \frac{3}{98} & 0 & -\frac{59}{98} & 0 & \frac{32}{49} & -\frac{4}{49} & & & \\ & & \frac{1}{12} & -\frac{2}{3} & 0 & \frac{2}{3} & -\frac{1}{12} & & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & & & & & & \end{bmatrix}$$

Accuracy

interior : 4

boundary: 2

overall: 3

$$H = \begin{bmatrix} \frac{17}{48} & & & & & & & & & & \\ & \frac{59}{48} & & & & & & & & & \\ & & \frac{43}{48} & & & & & & & & \\ & & & \frac{49}{48} & & & & & & & \\ & & & & 1 & & & & & & \\ & & & & & \ddots & & & & & \\ & & & & & & 1 & & & & \\ & & & & & & & \frac{49}{48} & & & \\ & & & & & & & & \frac{43}{48} & & \\ & & & & & & & & & \frac{59}{48} & \\ & & & & & & & & & & \frac{17}{48} \end{bmatrix}$$

SBP operators

$$
hD_6^{(1)} = \begin{bmatrix}
* & * & * & * & * & * & & & & & \\
* & * & * & * & * & * & & & & & \\
* & * & * & * & * & * & & & & & \\
* & * & * & * & * & * & * & & & & \\
* & * & * & * & * & * & * & * & & & \\
* & * & * & * & * & * & * & * & * & & \\
 & & & -\frac{1}{60} & \frac{3}{20} & -\frac{3}{4} & 0 & \frac{3}{4} & -\frac{3}{20} & \frac{1}{60} & \\
 & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{bmatrix}
$$

Accuracy

interior : 6

boundary: 3

overall: 4

$$
H = \begin{bmatrix}
* & & & & & & & \\
 & * & & & & & & \\
 & & * & & & & & \\
 & & & * & & & & \\
 & & & & * & & & \\
 & & & & & * & & \\
 & & & & & & 1 & \\
 & & & & & & & \ddots
\end{bmatrix}
$$

SBP operators

Theorem 7.1

For every $s>0$ there is a SBP operator $D^{(1)}$ with order of accuracy $2s$ in the interior and $2s-1$ near the boundary

Theorem 7.3

For $1 \leq s \leq 4$ there exists a SBP operator $D^{(1)}$ with interior and boundary order of accurcy $2s$ and s, with corresponding diagonal H

+ diagonal H is good for multi-D
- overall ccuracy is s+1: not optimal