Numerical Approximations of Partial Differential Equations in Theory and Practice

CME 325, winter 2008

Lecture 9

CME325, winter 08, Wednesday Feb 20

1. Initial-value problems (IVP), well-posedness and stability, periodic problems
2. Initial-boundary-value problems (IBVP), well-posedness and stability by energy estimates
3. Stability, Convergence and Accuracy
4. High order discretizations in time and in space
5. Coupled methods + how to measure accuracy
6. Errors: Damping, Dispersion, Conservation
7. Implementation of boundary conditions for hyperbolic IBVP
8. More applications of SBP+SAT
9. **Boundary conditions for parabolic IBVP+normal mode analysis**
10. Imbedded boundaries and interfaces
11. Non-reflecting boundary conditions

Parabolic IBVP

- SBP+SAT

- general technique using ghost points and extra boundary conditions that yields stable high order accurate methods.

General technique for Parabolic IBVP

Ex:

$u_t=u_{xx}$, $0\leq x\leq 1$, $t\geq 0$,

$u(0,t)=g_0(t)$,

$u_x(1,t)= g_1(t)$,

$u(x,0)=f(x)$,

Let $x_i, \quad i=-1,0,..N+1$ be equidistant points with

$x_0=0, \; x_{N\text{-}1}+x_N=1,$

$$\frac{du_i}{dt}=(D_+D_- - \tfrac{h^2}{12}D_+^2D_-^2)u_i, i=1,..N-1,$$

$u_0=g(t)$

$u_{-1}=?$

$D\_u_N=g_1(t)$, only order 2!

$u_{N+1}=?$

$u_i(0)=f_i$

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$$u_0=g(t)$$

$$D_+D_-u_0=g_0^{(1)}(t)+\frac{h^2}{12}g_0^{(2)}(t)$$

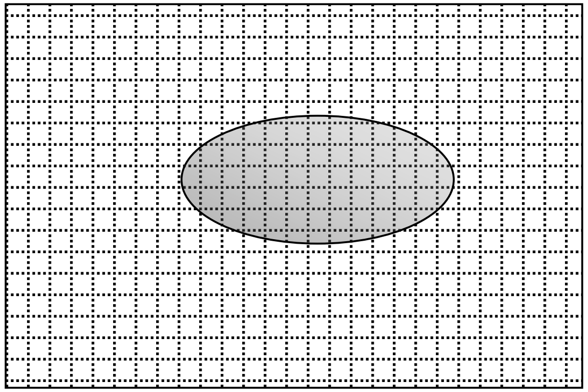
$$D_-u_N=g_1(t)+\frac{h^2}{24}g_1^{(1)}(t)$$

$$D_+D_-^2u_N=g_1^{(1)}(t)+\frac{h^2}{8}g_1^{(2)}(t)$$

$$u_i(0)=f_i$$

- The approximation is 4th order and semi-bounded (check this!)
- Generalizeable to higher order, multi-D and to systems

Embedded boundaries, preview

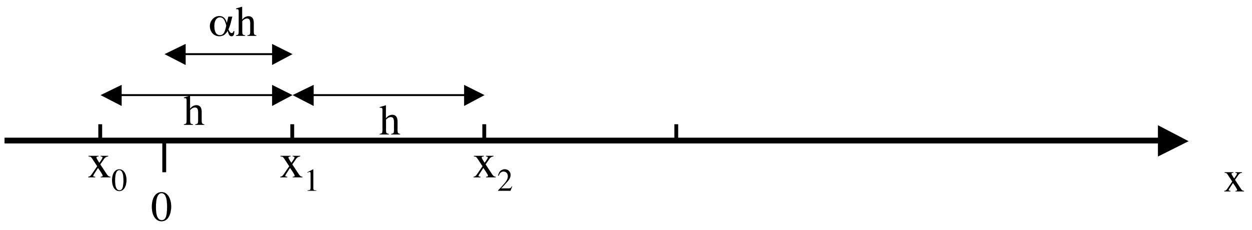


Grid is not aligned with boundary

+ no need for special meshes
-implementing boundary conditions is more difficult
- accuracy and stability difficult to achieve and analyze

Embedded boundary:1D example

$u_t+u_x=0$
$u(0,t)=g(t)$
$u(x,0)=f(x)$



$$\frac{du_j}{dt} = -D_0 u_j, \mathrm{j} = 1,..$$

$$u_0 = -\tfrac{\alpha}{1-\alpha} u_1$$

$$u_j(0) = f_j,$$

- Introduce a new grid point at x=0? Small cell leds to severe stability restrictions

- Use extrapolation to ghost point x_0? No general technique for stability and high order accuracy

Normal mode analysis

Cannot derive semi-boundedness?

Analyze each boundary seperatly by considering a half line problem with extra ”boundary condition” $||u||_h < \infty$

Look for solutions $\phi_j e^{st}$ where s is a complex number:

$$2sh\phi_j = -\phi_{j+1} + \phi_{j+1}, \ \mathrm{j} = 1, ..$$

$$\phi_0 = -\tfrac{\alpha}{1-\alpha}\phi_1$$

$$\|\phi\|_h < \infty$$

Eigenvalue problem for $\tilde{s} = sh$ and gridfunction ϕ

If there is an eigenvalue with Re $\tilde{s} > 0$ the discretization is unstable. (Why?)

Normal mode analysis

$$\tilde{s}\phi_j = hQ\phi_j, \mathrm{j} = 1,..$$

$$\mathrm{B_h}\phi_0 = 0, \ \|\phi\|_h < \infty$$

include only principle part!

Determinant condition:

- Assume *Re s>0*
- Determine general solution of difference equation satisfying $\| \ \|_h < \infty$
- Apply $\mathrm{B_h}$ yields C(s)σ=0
- ***Det C(s)≠0?***

Godunov - Ryabenkii condition is necessary for stability:

$\mathrm{Det\ C}(\tilde{\mathrm{s}}) \neq 0$ for all $\mathrm{Re}(\tilde{\mathrm{s}}) > 0$

Lemma: If Det $C(\tilde{s}) \neq 0$ for all Re $\tilde{\mathrm{s}} \geq 0$ then the approximation

is **boundary stable**: $\int_0^T |u_j(t)|^2 dt \leq K \int_0^T |g(t)|^2 dt$

Does boundary stability imply stability?

What if $\mathrm{Det\ C}(\tilde{\mathrm{s}}) \neq 0$ for some Re $\tilde{\mathrm{s}}_0 = 0$?

Normal mode analysis

Based on Laplace transform in time:

$$\hat{u}(s) = \int_0^\infty e^{-st} u(t)dt, \; s = \eta + i\xi,$$

$$s\hat{u}_j = Q\hat{u}_j + \hat{\mathrm{F}}_{\mathrm{j}}, \; j = 1,.. \quad , \mathrm{B}_{\mathrm{h}}\hat{u}_0 = \hat{g}, \; \|\hat{u}\|_h < \infty,$$

$$\int_0^\infty e^{-2\eta t} |u(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^\infty |\hat{u}(\eta + i\xi)|^2 d\xi,$$

Def : the discretization is **stable in the generalized sense** if for $g = f = 0$

$$\int_0^\infty e^{-2\eta t} \|u(t)\|_h^2 dt \le K(\eta) \int_0^\infty e^{-2\eta t} \|F(t)\|_h^2 dt, \quad \eta \ge \eta_0$$

Useful for convergence, can include boundary and initial data, lower order terms, two boundaries

Theorem 2.7: for symmetric hyperbolic problems boundary stability implies strong stability