Assignment 5

(Due in class March 14)

Consider the differential equation
\[ u_t = u_{xx} + F(x, t), \quad 0 \leq x \leq 2\pi, \quad 0 \leq t \leq 1, \]
\[ u(x, 0) = f(x), \]
where \( f(x) \) is 2\( \pi \)-periodic.

1. Find the solution for the case
\[ f(x) = \sin(\omega x), \]
\[ F(x, t) = (\omega^2 - \alpha) \sin(\omega x) e^{-\alpha t}, \]
where \( \omega \) and \( \alpha \) are constants. (Make the ansatz \( u(x, t) = a \sin(\omega x) e^{-bt} \).)

2. Discretize in space by using the 2nd and 4th order difference operators in Table 4.1, and also the 4th order Padé type operator in Table 4.5. Denote these operators by \( Q_1, Q_2, Q_3 \). Write a program that solves the problem by using the method
\[ u_{j}^{n+1} = (I + kQ_{\nu})u_{j}^{n} + kF_{j}^{n}, \quad \nu = 1, 2, 3. \]

3. Choose \( \omega = 5 \) and \( \alpha = 1 \) and run the program with the number of points in space determined by Table 1.2. Measure the error at \( t = 1 \) in the max-norm, and find out if there is good agreement with the first column in the table for \( Q_1 \) and \( Q_2 \). Does \( Q_3 \) give better results than \( Q_2 \)? If so, what is the explanation?

The method is only 1st order accurate in time, which means that one has to choose a small time step, for example \( k = 0.1h^2 \) for \( Q_1 \) and \( k = 0.01h^2 \) for \( Q_2 \) and \( Q_3 \).