Perturbed Determinants, Spectral Theory and Longest Cycles on Graphs

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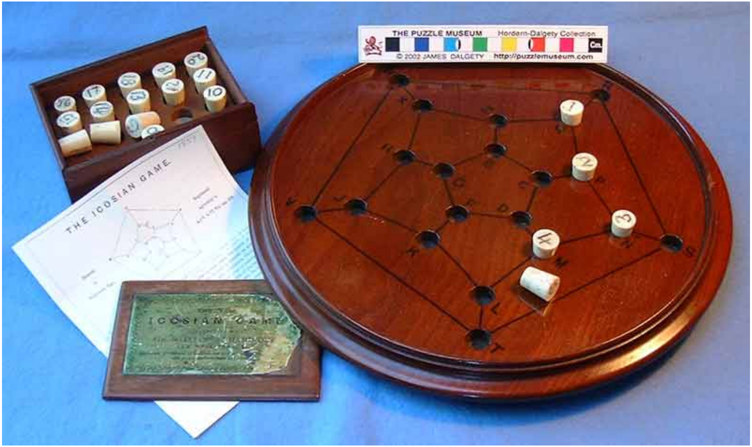
University of South Australia

UniSA and Flinders University, 8/11 and 10/11 2011

Emerging book

- V. Borkar, V. Ejov, J. A. Filar, G. Nguyen, Hamiltonian Cycle Problem and Markov Chains, to be published by Springer, 2012.

The Icosian game (1857)



Hamiltonian Cycles

- Hamiltonian Cycle (HC):

  A cycle that visits every vertex exactly once

A Hamiltonian Cycle

Another Hamiltonian Cycle

- Objective: Determine whether a graph possesses a HC

A Hamiltonian graph

A non-Hamiltonian graph

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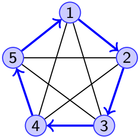

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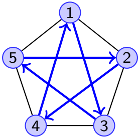

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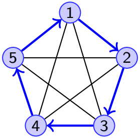

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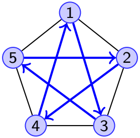

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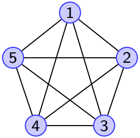

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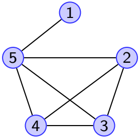

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Deterministic subgraphs

- Three types of deterministic subgraphs:

A Hamiltonian Cycle (HC):

$$P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A Short Cycle (SC):

$$P_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

A Noose Cycle (NC):

$$P_3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

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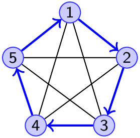

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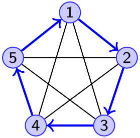

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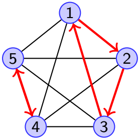

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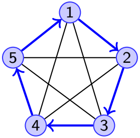

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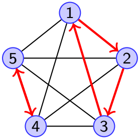

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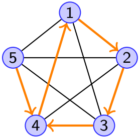

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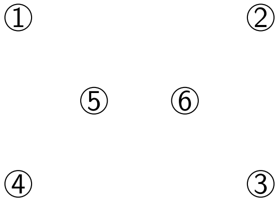
Hamiltonian cycle problem $\leftrightarrow$ Markov decision process

State space $\mathcal{S} = \{1, 2, \ldots, N\}$

Action space $\mathcal{A}(i)$

Policy f chooses action $a \in \mathcal{A}(i)$ at state i with probability f_{ia}

Probability transition matrix $\mathbf{P}(f)$



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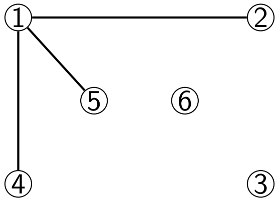

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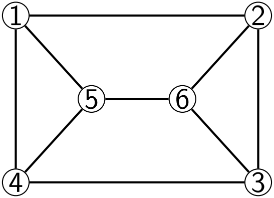

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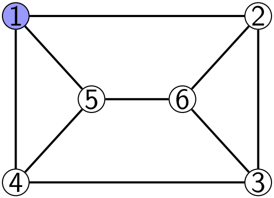

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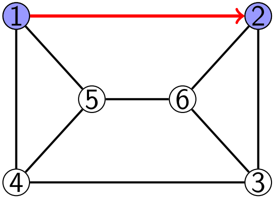

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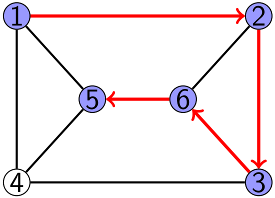

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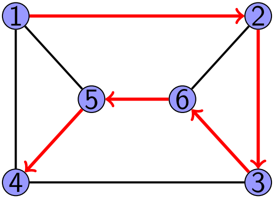

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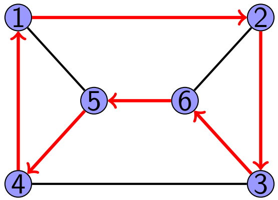

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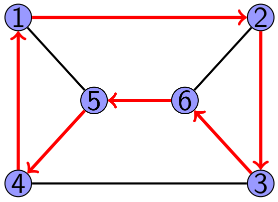
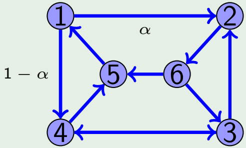


Hamiltonian cycle problem ↔ Markov decision process

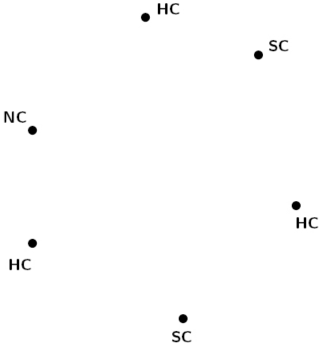
Example (Randomised Policy)

$$\mathbf{P} = \begin{bmatrix} 0 & \alpha & 0 & 1-\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1-\alpha & 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 & 1-\alpha & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\alpha & 0 & \alpha & 0 \end{bmatrix}$$



Deterministic Subgraphs and Randomised Chains

- For every graph, there is a set of deterministic subgraphs $\mathcal{D}$ (which might or might not contain Hamiltonian Cycles)



- Turn search space from discrete to continuous
- Randomised chains: From at least one vertex, there is a positive probability to go to two or more vertices.

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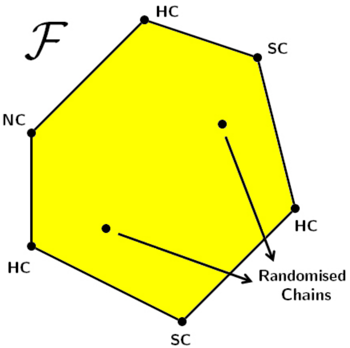

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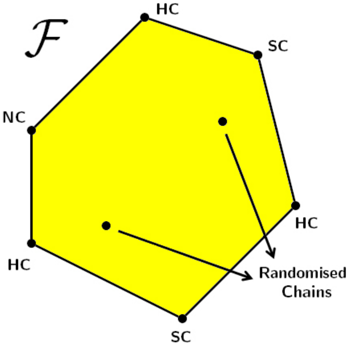

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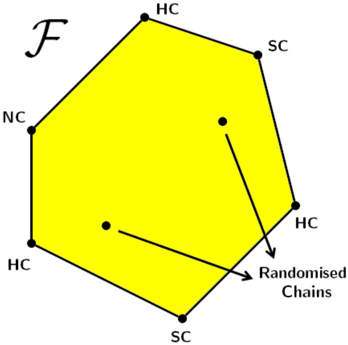

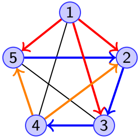

$$\mathbf{P}_4 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{6} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

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How to detect Hamiltonian Cycles?

Theorem (Borkar, Ejov & Filar, 2004)

Consider the symmetric linear perturbation:

$$\mathbf{P}_\varepsilon := (1-\varepsilon)\mathbf{P} + \frac{\varepsilon}{N}\mathbf{J}.$$

For $\mathbf{P}$ *in the space of all doubly stochastic policies* $\mathcal{DS}$

HCP is equivalent to
minimize $[(\mathbf{I} - \mathbf{P}_\varepsilon + \frac{1}{N}\mathbf{J})^{-1}]_{11}$ *over all* $\mathbf{P} \in \mathcal{DS}$.

Numerical difficulties and determinant function

- Numerical difficulties with optimisation algorithms
  - Hamiltonicity gap: $\Delta(N,\varepsilon) = \frac{3}{8N^2} - \mathcal{O}(\varepsilon)$
  - Evaluating inverses $[(\mathbf{I} - \mathbf{P}_\varepsilon + \frac{1}{N}\mathbf{J})^{-1}]_{11} := [\mathbf{A}^{-1}(\mathbf{P}_\varepsilon)]_{11}$
  - Dense Hessian matrices, due to perturbation
- Applying the adjoint form of the inverse:

$$[\mathbf{A}^{-1}(\mathbf{P}_\varepsilon)]_{11} = \frac{\det \mathbf{A}_{11}(\mathbf{P}_\varepsilon)}{\det \mathbf{A}(\mathbf{P}_\varepsilon)}$$

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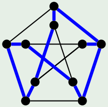
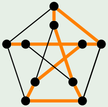
Deterministic determinant

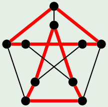
Non-Hamiltonian Graph of size $N = 10$



$\det \mathbf{A}(\mathbf{P}) = 9$



$\det \mathbf{A}(\mathbf{P}) = 8$



$\det \mathbf{A}(\mathbf{P}) = 0$

Deterministic determinant

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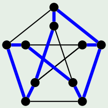

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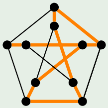

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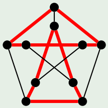

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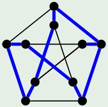

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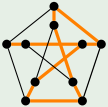

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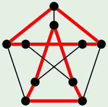

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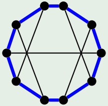
Hamiltonian Graph of size $N = 10$



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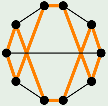
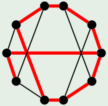


$\det \mathbf{A}(\mathbf{P}) = 10$



$\det \mathbf{A}(\mathbf{P}) = 8$

Maximum of Determinant

Theorem (Ejov, Filar, Murray & Nguyen)

For any graph G, any stochastic $\mathbf{P} \in \mathcal{F}$, and $\mathbf{A}(\mathbf{P}) := \mathbf{I} - \mathbf{P} + \frac{1}{N}\mathbf{J}$,

$$0 \leq \det \mathbf{A}(\mathbf{P}) \leq k,$$

where k is the length of the longest cycle in G.

Positive gap between non-Hamiltonian and Hamiltonian graphs

$$\left[\max_{\mathbf{P} \in \text{ non-Ham}} \det \mathbf{A}(\mathbf{P})\right] + 1 \leq \left[\max_{\mathbf{P} \in \text{ Ham}} \det \mathbf{A}(\mathbf{P})\right]$$

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Eigenvalue Inequality

Corollary (Ejov, Filar, Murray & Nguyen)

For any stochastic $N \times N$ *matrix* $\mathbf{P}$,

$$\prod_{i=1}^{N-1} (1 - \lambda_i) \leq N,$$

where λ_i*'s are the eigenvalues of* $\mathbf{P}$, *excluding* $\lambda_N = 1$.

Steps of the proof

$$0 \leq \det_{\mathbf{P} \in \mathcal{F}} \mathbf{A}(\mathbf{P}) = \prod_{i=1}^{N-1} (1 - \lambda_i) \leq k$$

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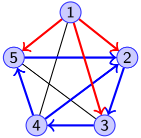
$$0 \le \det_{\mathbf{P}\in\mathcal{F}} \mathbf{A}(\mathbf{P}) = \prod_{i=1}^{N-1} (1-\lambda_i) \le k$$

Hamiltonian graph G $(N = k = 5)$

$$\det \mathbf{A}(\mathbf{P}) = \prod_{i=1}^{4} (1-\lambda_i) = 3.5 < 5$$

Consider all different cases:

- Randomised chains
- Deterministic subgraphs
  - Hamiltonian Cycles
  - Short Cycles
  - Noose Cycles



$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{6} & 0 & \frac{1}{3} \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Consider randomisation at each vertex i
Determinant is linear in each row i

Steps of the proof

$$0 \leq \det_{\mathbf{P}\in\mathcal{F}} \mathbf{A}(\mathbf{P}) = \prod_{i=1}^{N-1}(1-\lambda_i) \leq k$$

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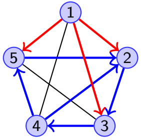

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$\det \mathbf{A}$(any randomised chain)
$\leq \det \mathbf{A}$(some deterministic subgraph)

Steps of the proof

$$0 \le \det_{\mathbf{P} \in \mathcal{F}} \mathbf{A}(\mathbf{P}) = \prod_{i=1}^{N-1} (1 - \lambda_i) \le k$$

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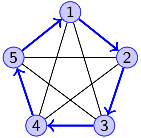

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

λ_i's are roots of unity
Elementary symmetric polynomials in λ_i

Steps of the proof

$$0 \leq \det_{\mathbf{P} \in \mathcal{F}} \mathbf{A}(\mathbf{P}) = \prod_{i=1}^{N-1} (1 - \lambda_i) \leq k$$

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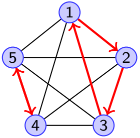
Hamiltonian graph G ($N = k = 5$)

$$\det \mathbf{A}(\mathbf{P}) = \prod_{i=1}^{4} (1 - \lambda_i) = 0$$



$$\left[\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Construct a special eigenvector
Eigenvalue of unity is of multiplicity ≥ 2

Steps of the proof

$$0 \leq \det_{\mathbf{P} \in \mathcal{F}} \mathbf{A}(\mathbf{P}) = \prod_{i=1}^{N-1} (1 - \lambda_i) \leq k$$

Hamiltonian graph G $(N = k = 5)$

$$\det \mathbf{A}(\mathbf{P}) = \prod_{i=1}^{4} (1 - \lambda_i) = 4$$

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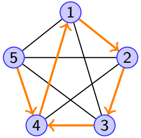

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Change bases of $\mathbb{R}^N$ from **E** to **V**
Principal leading minor of $\mathbf{B} := \mathbf{P} - 1/N\mathbf{J}$

Maximum of Determinant - Symmetric Linear Perturbation

Theorem (Ejov & Nguyen)

Consider the symmetric linear perturbation:

$$\mathbf{P}_\varepsilon := (1-\varepsilon)\mathbf{P} + \frac{\varepsilon}{N}\mathbf{J}.$$

For any graph G, any stochastic $\mathbf{P} \in \mathcal{F}$, and $\varepsilon \in [0,1)$

$$\begin{aligned} 0 \leq \det \mathbf{A}(\mathbf{P}_\varepsilon) &\leq \frac{1-(1-\varepsilon)^k}{\varepsilon} \\ &= k + \varepsilon(\cdots) + \varepsilon^2(\cdots) + \mathcal{O}(\varepsilon^3), \end{aligned}$$

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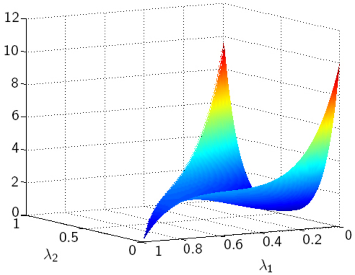
- This demonstrates the robustness of the determinant function.

An Optimisation Problem

HCP is equivalent to
maximise $\det \mathbf{A}(\mathbf{P})$ over all $\mathbf{P} \in \mathcal{F}$.

- Non-linear & non-concave
- Narrow search space from $\mathcal{F}$ to $\mathcal{DS}$
- Empirically, in $\mathcal{DS}$ no interior max or min



Graph of $\det \mathbf{A}(\mathbf{P}_\lambda)$,

where $\mathbf{P}_\lambda = \lambda_1 \mathbf{P}_1 + \lambda_2 \mathbf{P}_2 + \lambda_3 \mathbf{P}_3$.

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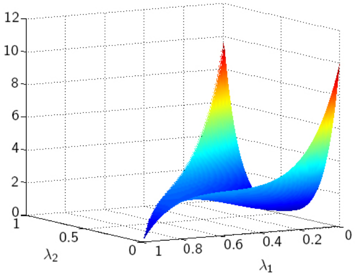

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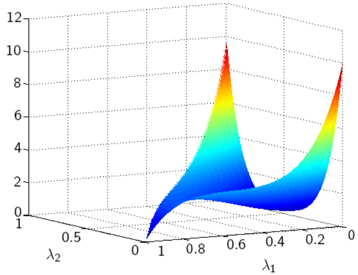

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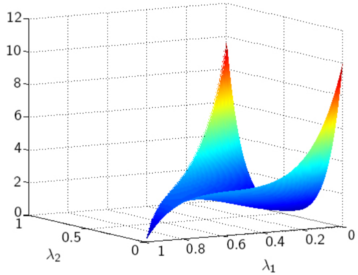

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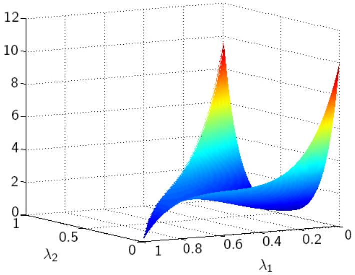

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Trace of the Fundamental matrix

Fundamental matrix: $\mathbf{G}(\mathbf{P},\varepsilon) := (\mathbf{I} - \mathbf{P}_\varepsilon + \mathbf{P}^*(\mathbf{P}_\varepsilon))^{-1}$

Theorem (Litvak, Ejov)

For $\varepsilon \in [0,1)$ the minimizers of $\mathrm{Tr}[\mathbf{G}(\mathbf{P},\varepsilon)]$ over the set of all doubly stochastic policies correspond to Hamiltonian cycles.

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For $\varepsilon = 0$ and for any stochastic policy $\mathbf{P}$ feasible on a given Hamiltonian graph,

$$\min_{\mathbf{P}} \mathrm{Tr}[\mathbf{G}(\mathbf{P},\varepsilon)] = \mathrm{Tr}[\mathbf{G}\left(\mathbf{P}_{HC},\varepsilon\right)],$$

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Hamiltonicity Trace conjecture

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Trace conjecture solution

Theorem (Litvak, Nguyen, Taylor, Ejov (2011))

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for any $\mathbf{P}_{HC}$ corresponding to a Hamiltonian cycle.

Idea of the proof

- Derive relationships between eigenvalues and eigenvectors of various relevant matrices, that lead to the derivation of formulae for the trace function;
- Prove that the value of the trace of $\mathbf{G}(\mathbf{P}, \varepsilon)$, for any randomized policy $\mathbf{P}$ and $\varepsilon \in (0, 1)$, is bounded below (and, also, above) by that of some deterministic policy;
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Snakes and Ladders heuristic (SLH)

Dirac-Pósa theorem

Theorem (Dirac, Pósa)

A graph with N vertices ($N \geq 3$) in which every vertex has a degree at least $\frac{N}{2}$ is Hamiltonian.

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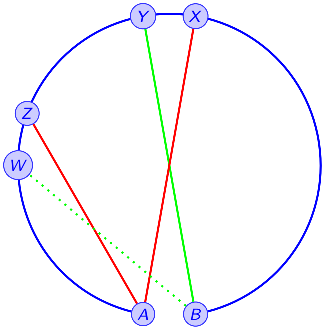
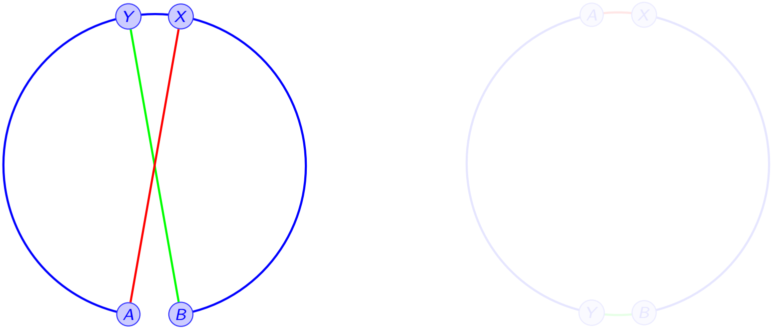


Illustration of the proof



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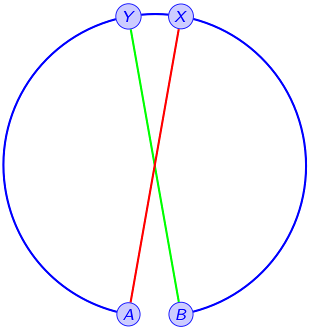
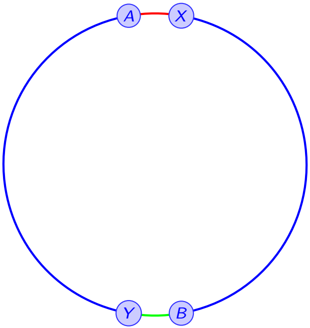
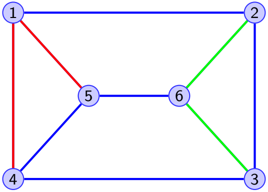




Graph circular form



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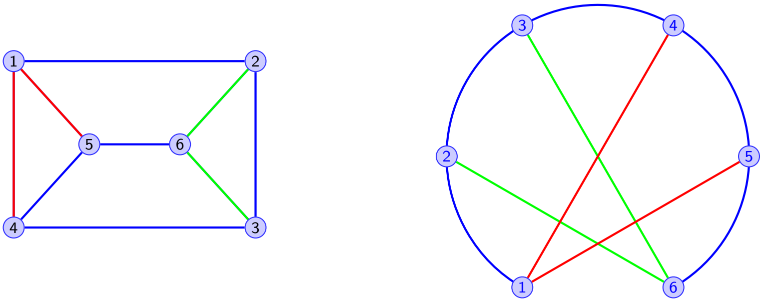
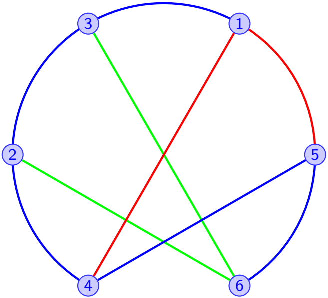


Hamiltonian cycle



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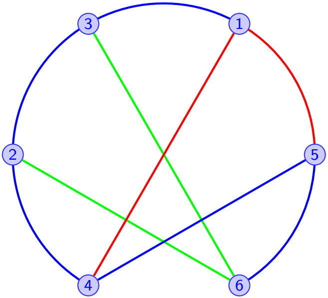

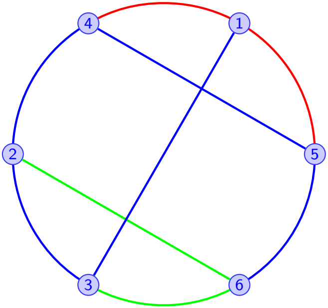

THANK YOU!