BARON: Branch and Reduce Optimization Navigator

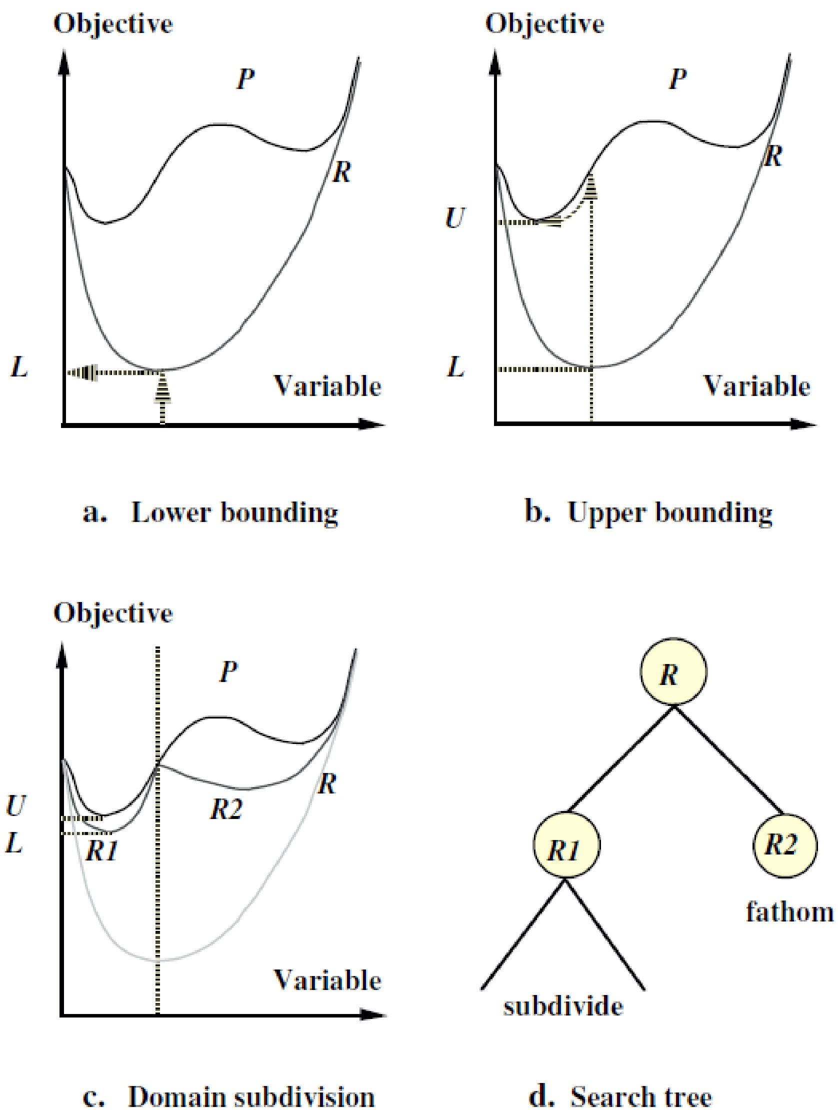
A High Level Overview for CME334

Assembled by Thomas Lipp

What is BARON?

- BARON is a branch and bound non-linear, mixed integer global optimization solver dating back to 1991, although last updated in 2010
- Developed at University of Illionis at Urbana-Champaign by Nikolaos Sahinidis.
- Problems can be stated in either the AIMMS or GAMS modeling languages, although computing time is available on the NEOS server for GAMS
- Objective and constraint functions must be factorable
- All non-linear variables and expressions must be bounded from above and below

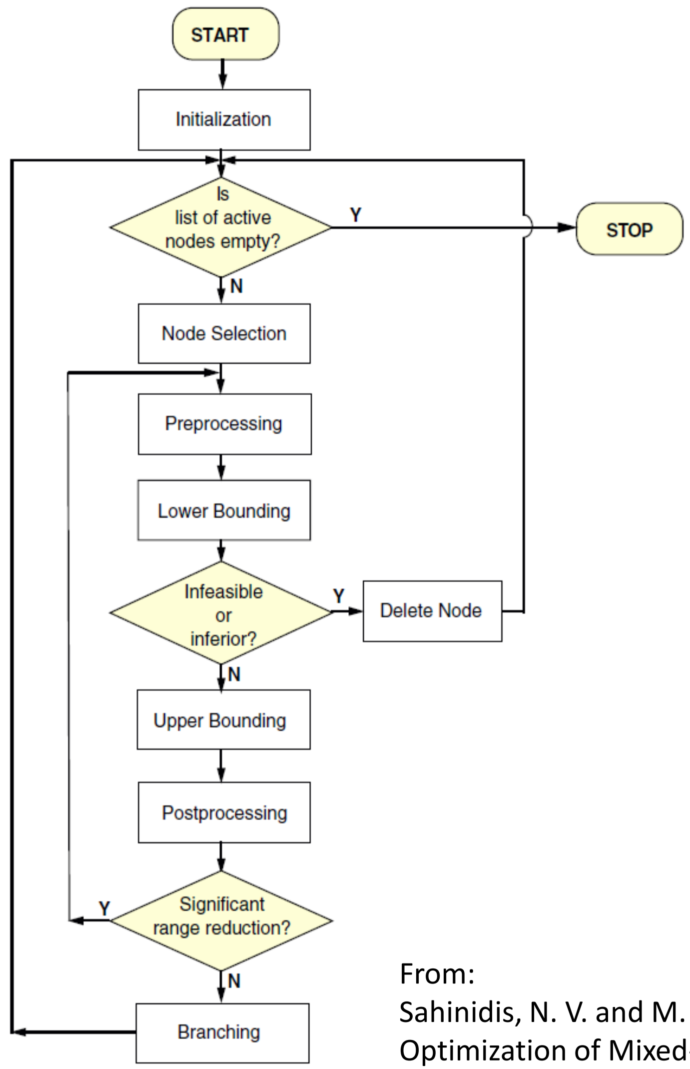
Branch and Bound Algorithms



A quick reminder of Branch and Bound algorithms.

From:
Sahinidis, N. V. and M. Tawarmalani, BARON 9.0.4: Global Optimization of Mixed-Integer Nonlinear Programs, User's manual, 2010

What does the Baron Algorithm Look like?



From:
Sahinidis, N. V. and M. Tawarmalani, BARON 9.0.4: Global Optimization of Mixed-Integer Nonlinear Programs, User's manual, 2010

What does each step do?

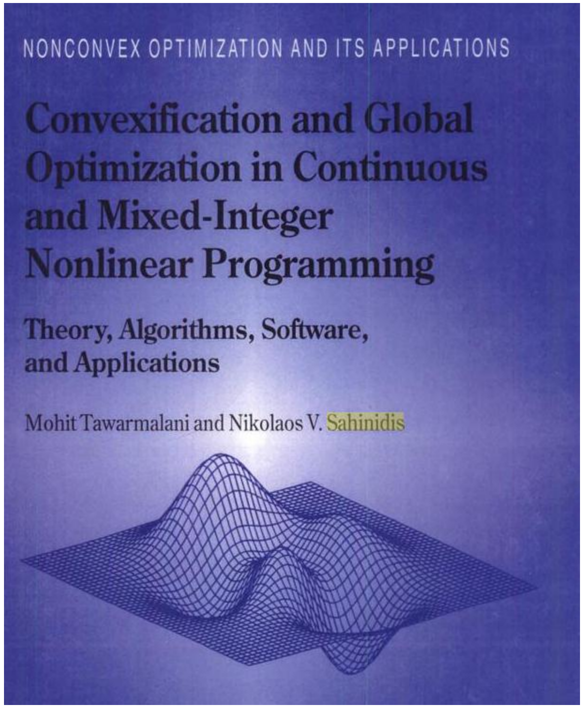
- Unfortunately, the Baron documentation does not address each step fully. BARON is written to be modular so that user subroutines can be used to replace certain steps as needed.
- The creators are also trying to market BARON: http://www.theoptimizationfirm.com/

The Optimization Firm

- The subproblems are generally fed to existing solvers: SNOPT, MINOS, CPLEX, etc.

Things I won't be able to answer

- How does BARON handle mixed integers
- What exactly occurs in any given step
- There may be more information available in:

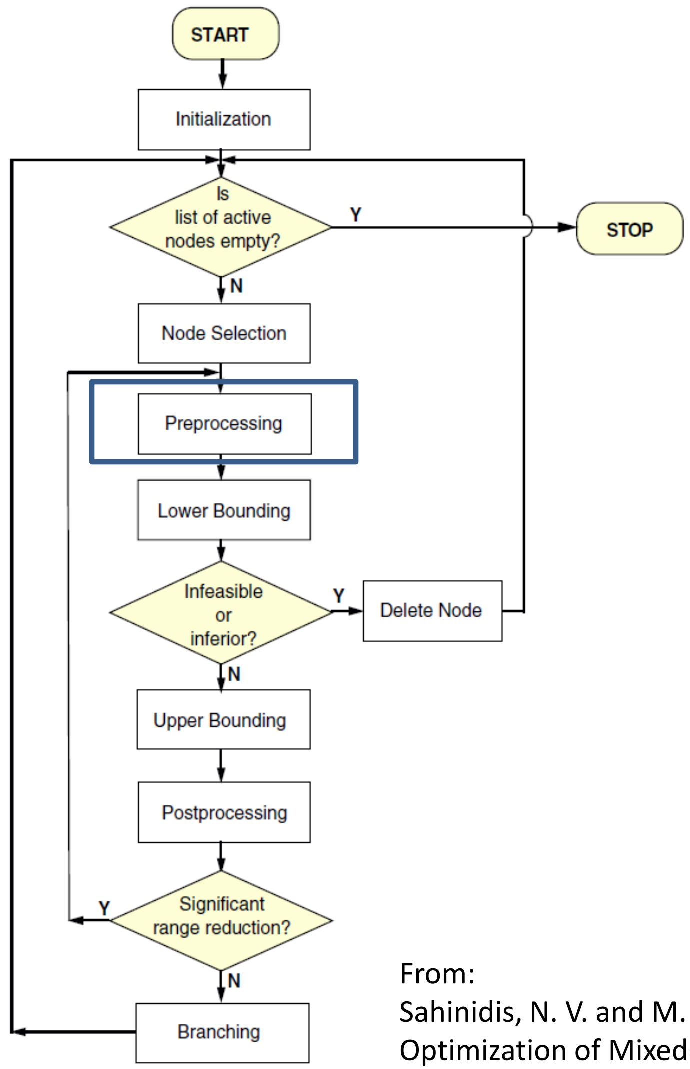


Available for $221.09, and not currently owned by the Stanford Library.

What does BARON have to offer, then?

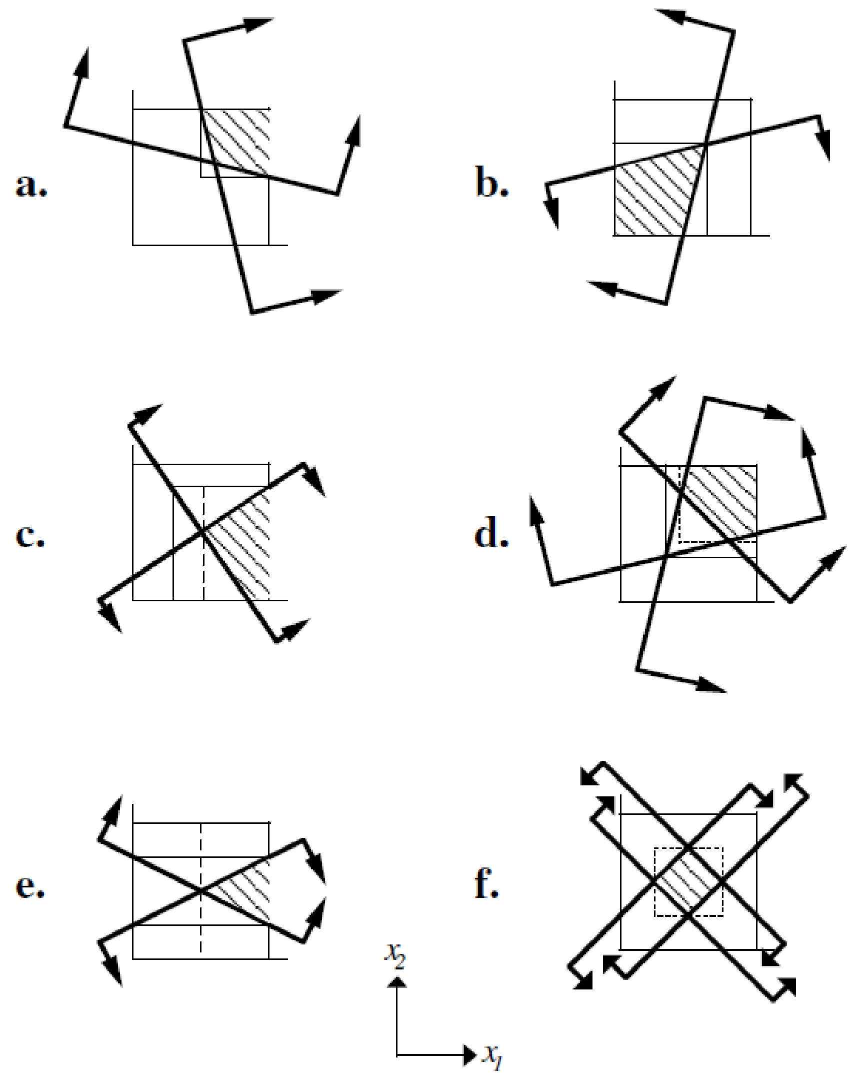
- BARON calls itself a branch and reduce algorithm rather than a branch and bound algorithm, and it is this “reducing” that will provide the most insight.
- The modular nature of BARON allows it to be tailored to your specific application.

Preprocessing on a Node



From:
Sahinidis, N. V. and M. Tawarmalani, BARON 9.0.4: Global Optimization of Mixed-Integer Nonlinear Programs, User's manual, 2010

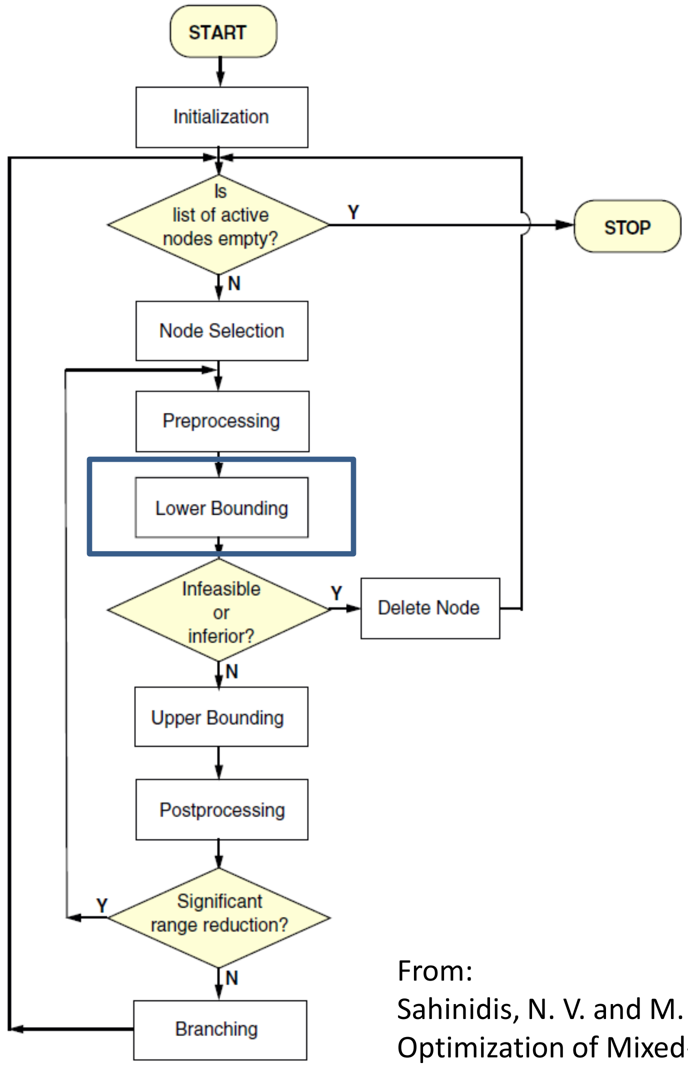
Feasibility Based Range Reduction: a “Poor Man’s LP”



- The solid outer box represents the initial domain of the node.
- The solid inner box represents the reduced domain by considering each of the constraints individually. “The Poor Man’s LP”
- The dotted boundary represents solving for the limits on each coordinate with all constraints active
- The shaded region is the true feasible region.

From:
Sahinidis, N. V. and M. Tawarmalani, BARON 9.0.4: Global Optimization of Mixed-Integer Nonlinear Programs, User's manual, 2010

Lower Bounding



From:
Sahinidis, N. V. and M. Tawarmalani, BARON 9.0.4: Global Optimization of Mixed-Integer Nonlinear Programs, User's manual, 2010

Lower Bounding

This is an overview of the lower bounding scheme used by BARON, but is a little complex to go over. Remember that one requirement of all of our functions was that they be factorable:

Algorithm Relax f(x)

If $f(x)$ is a function of single variable $x \in [x^l, x^u]$, then
  Construct under- and over-estimators for $f(x)$ over $[x^l, x^u]$,
else if $f(x) = g(x)/h(x)$, then
    Fractional_Relax (f, g, h)
  end of if
else if $f(x) = \prod_{i=1}^{t} f_i(x)$, then
  for $i := 1$ to t do
    Introduce variable y_{f_i}, such that $y_{f_i} = \text{Relax } f_i(x)$
  end of for
  Introduce variable y_f, such that $y_f = \text{Multilinear\_Relax } \prod_{i=1}^{t} y_{f_i}$
else if $f(x) = \sum_{i=1}^{t} f_i(x)$, then
  for $i := 1$ to t do
    Introduce variable y_{f_i}, such that $y_{f_i} = \text{Relax } f_i(x)$
  end of for
  Introduce variable y_f, such that $y_f = \sum_{i=1}^{t} y_{f_i}$
else if $f(x) = g(h(x))$, then
  Introduce variable $y_h = \text{Relax } h(x)$
  Introduce variable $y_f = \text{Relax } g(y_h)$
end of if

From:
M. Tawarmalani and N. V. Sahinidis (2004), 'Global optimization of mixed-integer
nonlinear programs: A theoretical and computational study', Math. Program., DOI 10.1007/s10107-003-0467-6

Note: there are other bounding schemes discussed, but this appears to be the one that is used

Lower Bounding Subroutines

Algorithm Multilinear_Relax $\prod_{i=1}^{t} \mathbf{y}_{\mathbf{r_i}}$ (Recursive Arithmetic)

for $i := 2$ to t do
  Introduce variable $y_{r_1,\dots,r_i}$ = Bilinear_Relax $y_{r_1,\dots,r_{i-1}} y_{r_i}$.
end of for

Algorithm Bilinear_Relax $\mathbf{y_i y_j}$ (Convex/Concave Envelope)

$$\begin{array}{lcl}
\text{Bilinear_Relax } y_i y_j & \geq & y_i^u y_j + y_j^u y_i - y_i^u y_j^u \\
\text{Bilinear_Relax } y_i y_j & \geq & y_i^l y_j + y_j^l y_i - y_i^l y_j^l \\
\text{Bilinear_Relax } y_i y_j & \leq & y_i^u y_j + y_j^l y_i - y_i^u y_j^l \\
\text{Bilinear_Relax } y_i y_j & \leq & y_i^l y_j + y_j^u y_i - y_i^l y_j^u
\end{array}$$

Algorithm Fractional_Relax (f, g, h) (Product Disaggregation)

Introduce variable y_f
Introduce variable y_g = Relax $g(x)$
if $h(x) = \sum_{i=1}^{t} h_i(x)$, then
  for $i := 1$ to t do
    Introduce variable y_{f,h_i} = Relax $\left(y_f h_i(x)\right)$
  end of for
  Introduce relation $y_g = \sum_{i=1,\dots,t} y_{f,h_i}$
else
  Introduce variable y_h = Relax $h(x)$
  Introduce relation y_g = Bilinear_Relax $y_f y_h$
end of if

Algorithm Univariate_Relax $\mathbf{f(x_j)}$ (Recursive Sums and Products)

if $f(x_j) = cx_j^p$ then
  Introduce variable y_f = Monomial_Relax cx_j^p
else if $f(x_j) = cp^{x_j}$ then
  Introduce variable y_f = Power_Relax cp^{x_j}
else if $f(x_j) = c\log(x_j)$ then
  Introduce variable y_f = Logarithmic_Relax $c\log(x_j)$
else if $f(x_j) = \prod_{i=1}^{t} f_i(x_j)$, then
  for $i := 1$ to t do
    Introduce variable y_{f_i}, such that y_{f_i} = Relax $f_i(x)$
  end of for
  Introduce variable y_f, such that y_f = Multilinear_Relax $\prod_{i=1}^{t} y_{f_i}$
else if $f(x_j) = \sum_{i=1}^{t} f_i(x_j)$, then
  for $i := 1$ to t do
    Introduce variable y_{f_i}, such that y_{f_i} = Relax $f_i(x_j)$
  end of for
  Introduce variable y_f, such that $y_f = \sum_{i=1}^{t} y_{f_i}$
end of if

Lower Bounding Bilinearities

$$x_i x_j < 0$$

$$(x_i^U - x_i)(x_j^U - x_j) \geq 0$$

$$x_i^U x_j^U - x_i^U x_j - x_i^U x_j + x_i x_j > 0$$

$$x_i x_j > x_i^U x_j + x_i^U x_j - x_i^U x_j^U$$

$$x_i x_j > x_i^L x_j + x_i^L x_j - x_i^L x_j^L$$

So create a new variable to replace the bilinearity

$$w_{ij} < 0$$

$$w_{ij} > x_i^U x_j + x_i^U x_j - x_i^U x_j^U$$

$$w_{ij} > x_i^L x_j + x_i^L x_j - x_i^L x_j^L$$

A similar procedure can be performed in the case $x_i x_j > 0$
and for upper bounding.

Lower Bounding Linear Fractional Terms

- The same inequalities used in the bilinear case can be rearranged, with some added technicalities on sign to produce the liner fractional inequalities:

$$\frac{x_i}{x_j} \geq \frac{x_i^U}{x_j} + \frac{x_i}{x_j^L} - \frac{x_i^U}{x_j^L}$$

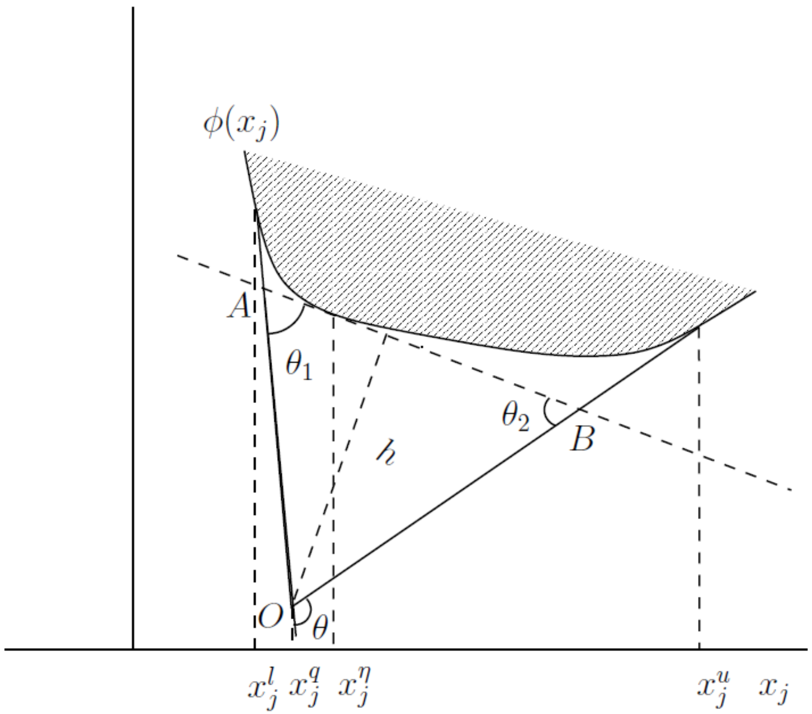
$$\frac{x_i}{x_j} \geq \frac{x_i^L}{x_j} + \frac{x_i}{x_j^U} - \frac{x_i^L}{x_j^U}$$

$$\frac{x_i}{x_j} \leq \frac{x_i^U}{x_j} + \frac{x_i}{x_j^U} - \frac{x_i^U}{x_j^U}$$

$$\frac{x_i}{x_j} \leq \frac{x_i^L}{x_j} + \frac{x_i}{x_j^L} - \frac{x_i^L}{x_j^L}$$

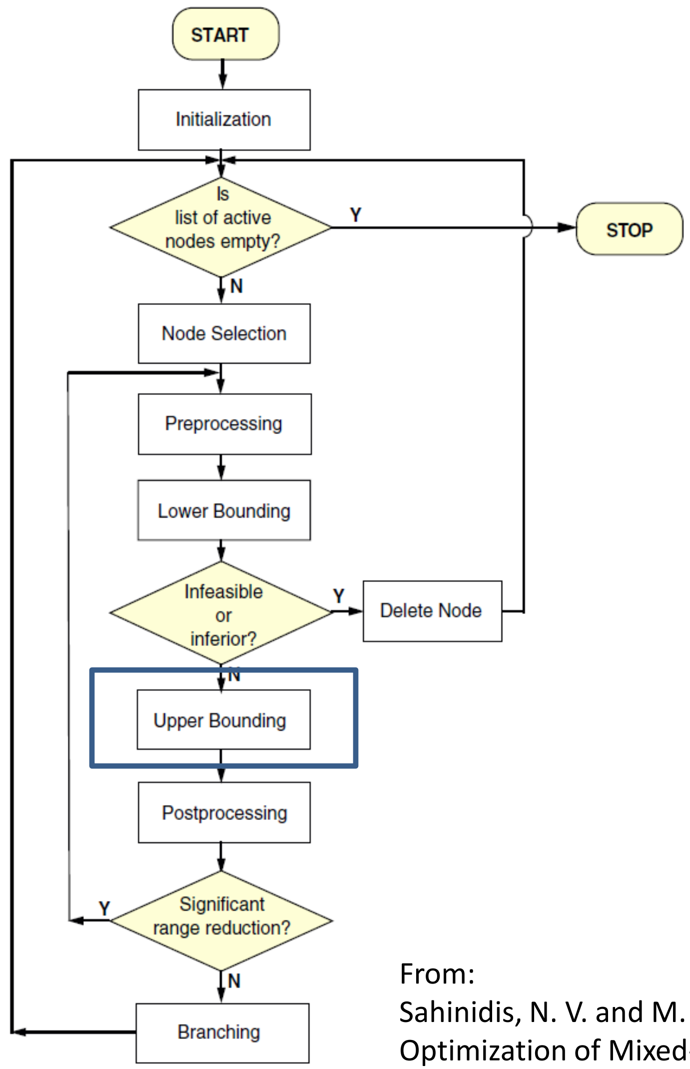
Further Lower Bounding

- Once a lower bound is created, a linear approximation of this bound will often be created to further speed computation.



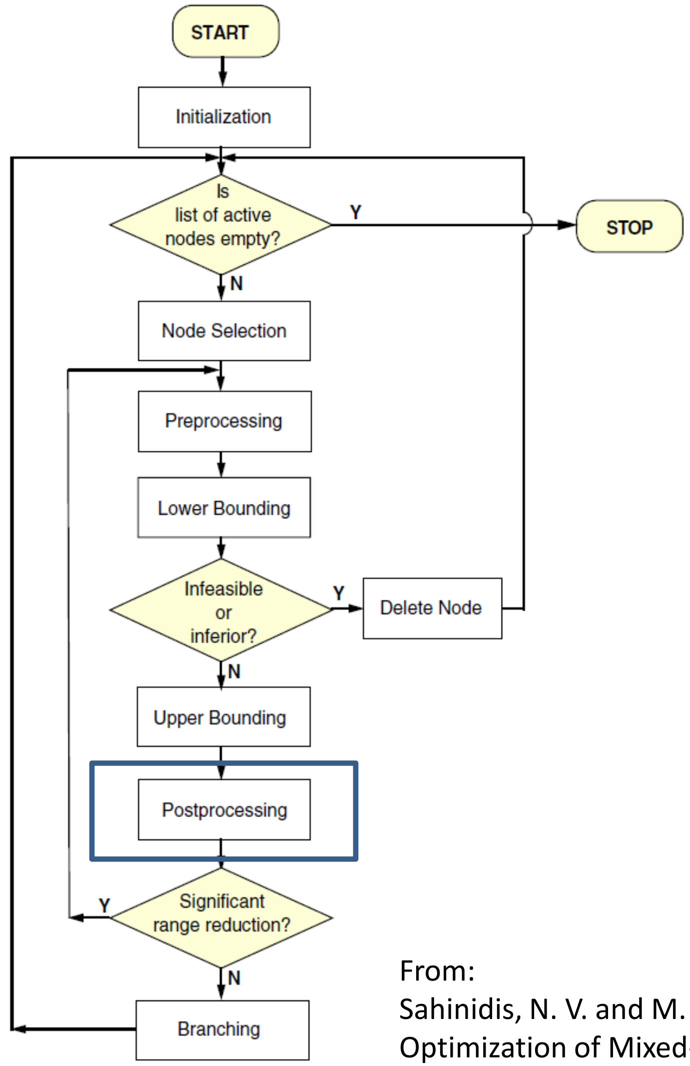
From:
M. Tawarmalani and N. V. Sahinidis (2004), 'Global optimization of mixed-integer
nonlinear programs: A theoretical and computational study', Math. Program., DOI 10.1007/s10107-003-0467-6

Upper Bounding



From:
Sahinidis, N. V. and M. Tawarmalani, BARON 9.0.4: Global Optimization of Mixed-Integer Nonlinear Programs, User's manual, 2010

Post Processing on a Node



From:
Sahinidis, N. V. and M. Tawarmalani, BARON 9.0.4: Global Optimization of Mixed-Integer Nonlinear Programs, User's manual, 2010

Optimality Based Range Reduction

- We will first define three different problems:

$$
(P): \quad \begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g(x) \leq 0 \\ & x \in X \end{aligned}
$$

Original problem

$$
(R): \quad \begin{aligned} \min \quad & \bar{f}(\bar{x}) \\ \text{s.t.} \quad & \bar{g}(\bar{x}) \leq 0 \\ & \bar{x} \in \bar{X} \end{aligned}
$$

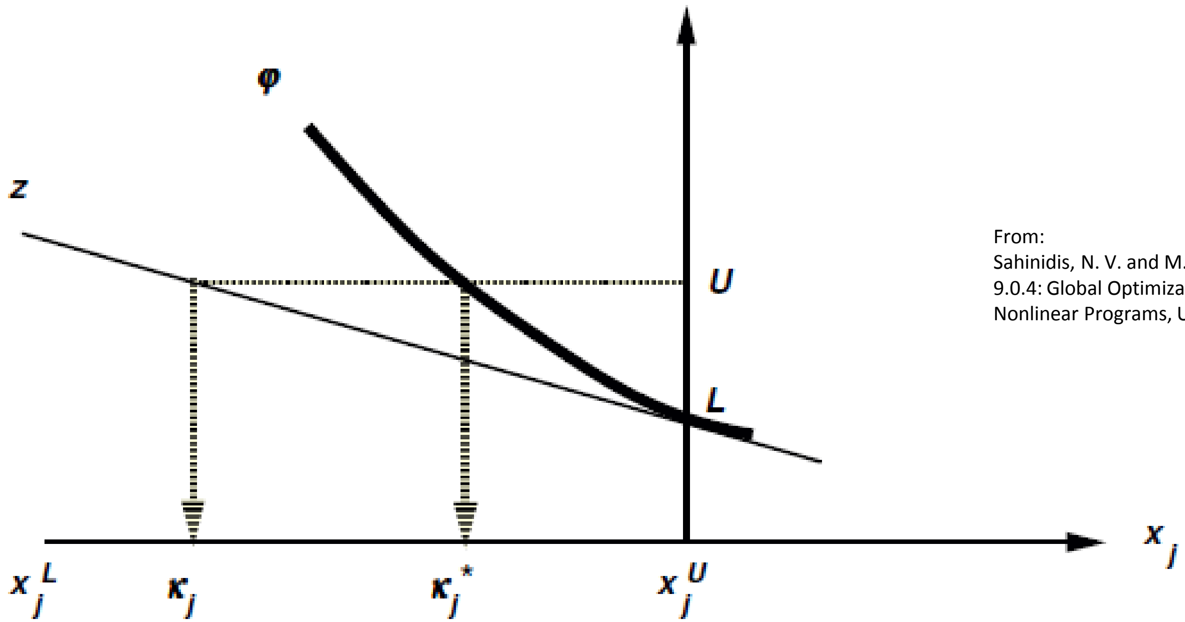
Relaxed problem

$$
(R_y): \quad \varphi(y) = \begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g(x) \leq y \\ & x \in X \end{aligned}
$$

Perturbed problem

Optimality Based Range Reduction: on an active constraint

In this example the upper bound range constraint is active on x



From:
Sahinidis, N. V. and M. Tawarmalani, BARON 9.0.4: Global Optimization of Mixed-Integer Nonlinear Programs, User's manual, 2010

L: the solution of the relaxed problem R

U: an upper bound on the original problem P

Φ: Φ(y) unknown solution of R(y) but is L when y = 0, as R(0) = R.

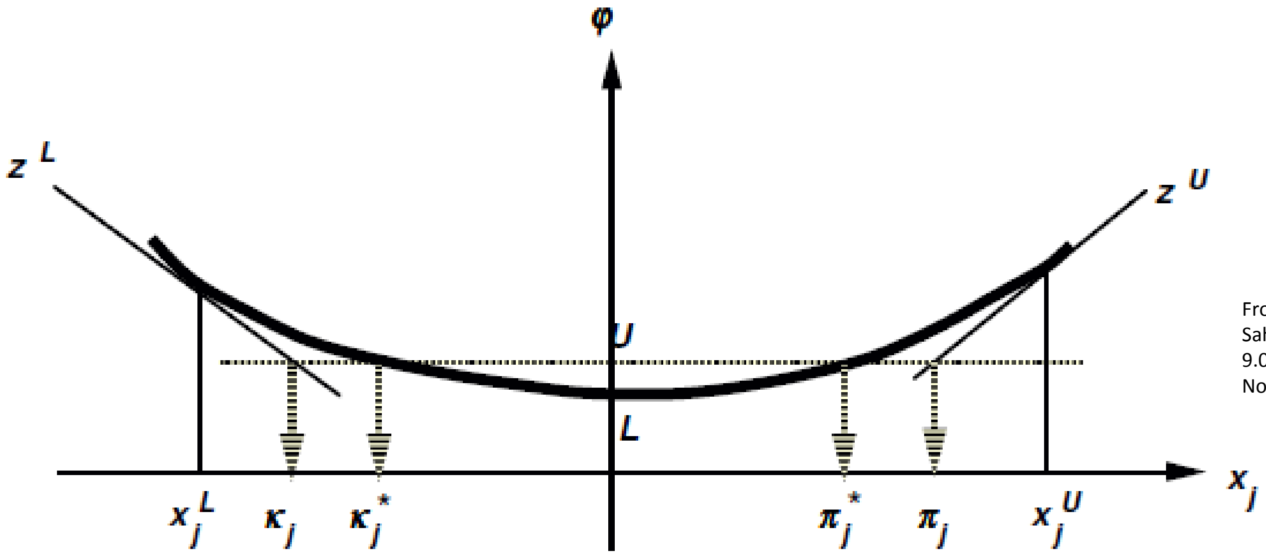
K_j*: the range reduction if Φ(y)

K_j: the range reduction based on the supporting hyperplane of Φ(y), z

Z: a supporting hyperplane based on the Lagrange multiplier in the solution of R.

Optimality Based Range Reduction: on an inactive constraint

In this example there is no range constraint active.



From:
Sahinidis, N. V. and M. Tawarmalani, BARON 9.0.4: Global Optimization of Mixed-Integer Nonlinear Programs, User's manual, 2010

L: the solution of the relaxed problem R

U: an upper bound on the original problem P

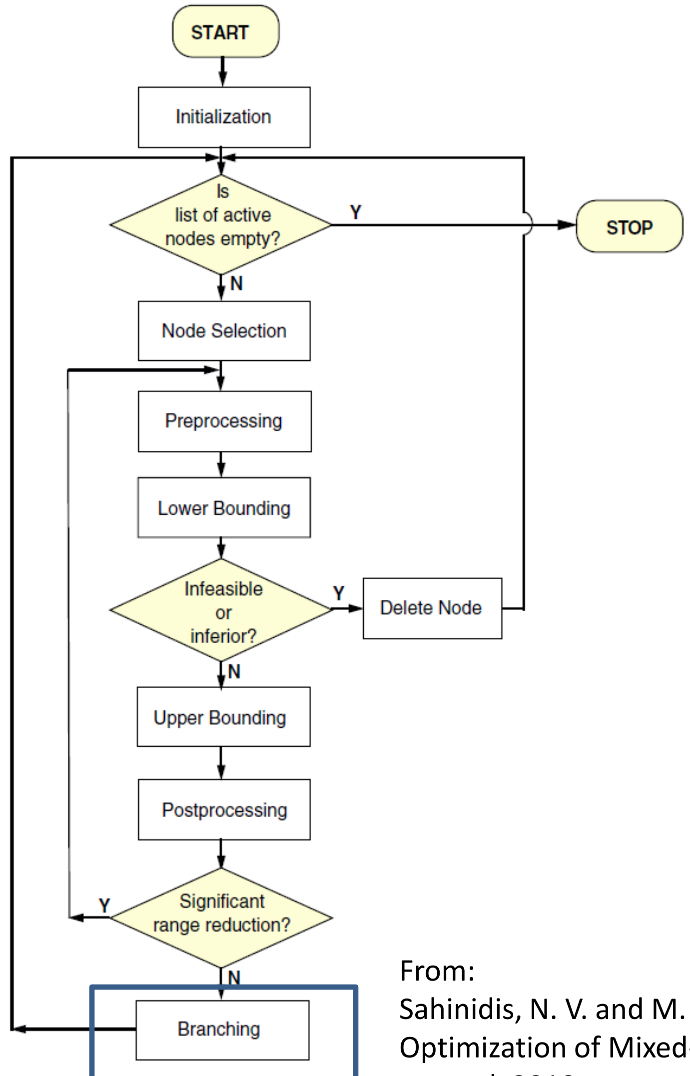
z^L: linear underestimator found by fixing the value of x to be x_j^L

z^U: linear underestimator found by fixing the value of x to x_j^U

K_j^*/π_j^*: the range reduction if the entire value function were known.

K_j/π_j: the range reduction based on the underestimators z.

Branching



From:
Sahinidis, N. V. and M. Tawarmalani, BARON 9.0.4: Global Optimization of Mixed-Integer Nonlinear Programs, User's manual, 2010

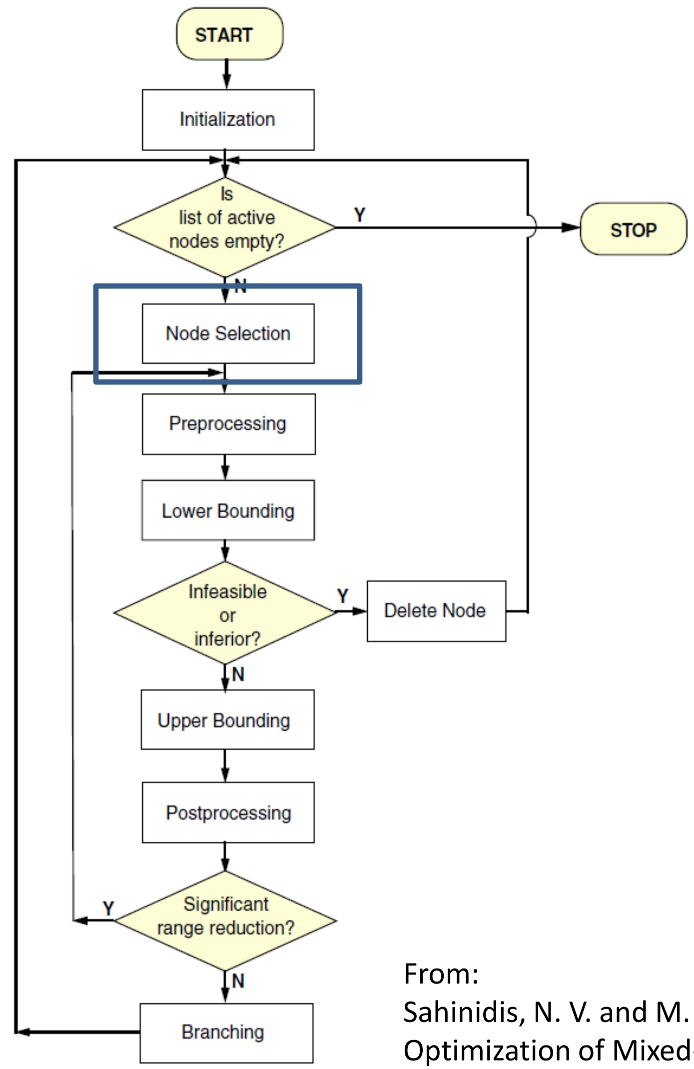
Branching: Variable Selection

- BARON uses a rectangular subdivision scheme, so only a single variable is chosen for each branching.
- The variable chosen to branch is the variable that contributes the most to the “relaxation gap”.
- This is a non trivial process as many additional variables were introduced in our underestimating step

Branching: Point Selection

- Periodically, branch at the midpoint
- Otherwise, branch at the solution of the lower bounding problem.
- This decision is stored, but not executed until the node is selected again.

Node Selection



From:
Sahinidis, N. V. and M. Tawarmalani, BARON 9.0.4: Global Optimization of Mixed-Integer Nonlinear Programs, User's manual, 2010

Node Selection

- Default BARON implementation is a composite value based on lower bound, violation (sum of violations of all variables) and order of creation.
- If memory limits are approached BARON switches to FIFO.
- As with all steps, customizable by the user.

For More Information

- Sahinidis, N. V. and M. Tawarmalani, BARON 9.0.4: Global Optimization of Mixed-Integer Nonlinear Programs, User's manual, 2010
- M. Tawarmalani and N. V. Sahinidis , ‘Global optimization of mixed-integer nonlinear programs: A theoretical and computational study, Math Program, DOI 10.1007/s10107-003-0467-6, 2004
- M. Tawarmalani and N. V. Sahinidis, Convexification and global optimization in continuous and mixed-integer nonlinear programming: theory, algorithms, software, and applications, Springer, 2002