Inverse Finite Element Problems in Modeling the Corneal Swelling

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Objective

- Corneal swelling is critical for the health of the eye.
- To develop reasonable models to predict corneal swelling behavior under normal/pathological conditions.
- Obtain the material properties via inverse finite element approach.

Motivation

Model parameters
(material properties)

Nonlinear coupled PDEs

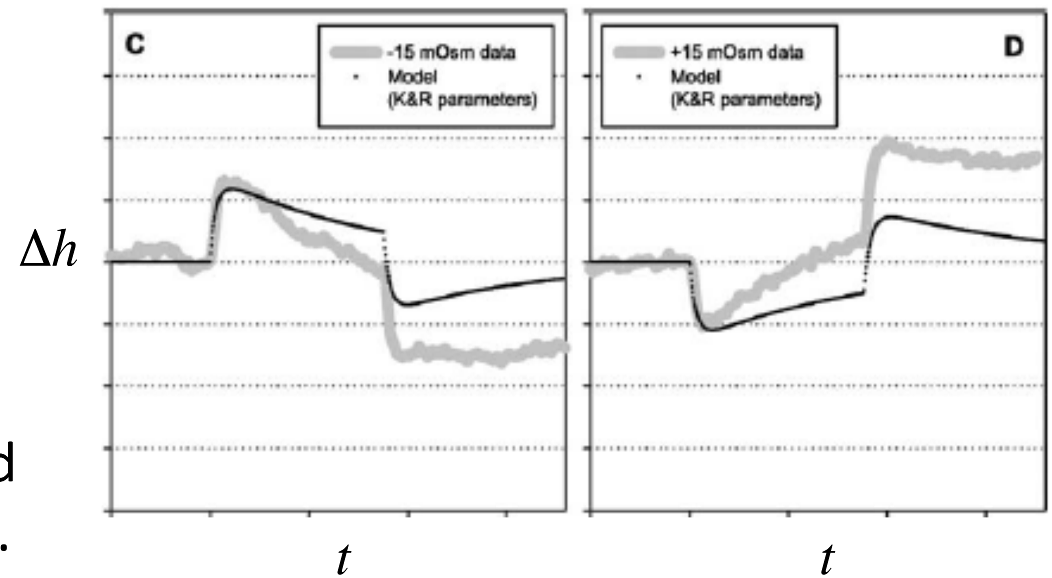
$$\frac{\partial W_f}{dt} = -\frac{\partial}{\partial \xi}\left(\frac{\rho_f}{\rho_s} J_f\right), \qquad \frac{\partial}{dt}\left(\frac{W_f C_k}{\rho_f}\right) = -\frac{\partial}{\partial \xi}\left(\frac{J_k}{\rho_s}\right)$$

Experimentally measured quantity

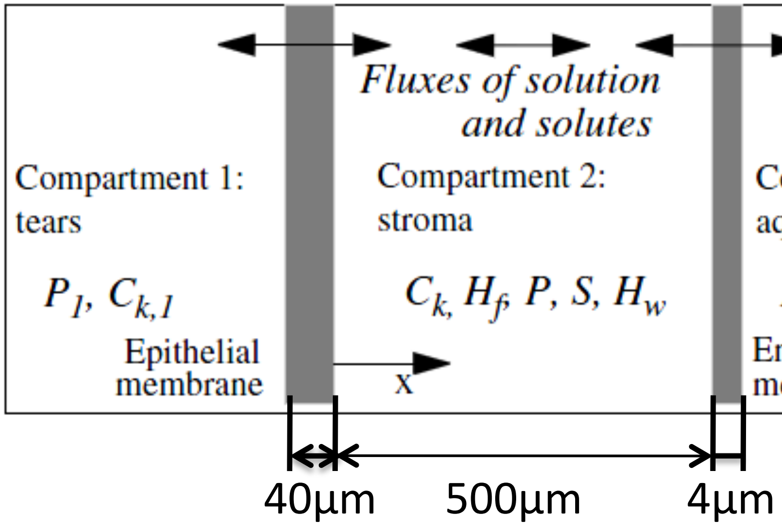
$$\bar{W} = \frac{1}{V}\int_{\Omega} W_f\big|_{t\to\infty}\, d\Omega$$

$$\Delta h(t) = \int_0^{\xi_{\max}} \left(\frac{W_f(t,x)}{\rho_f(t,x)} - \frac{W_{f0}}{\rho_{f0}}\right)\rho_d \, d\xi$$

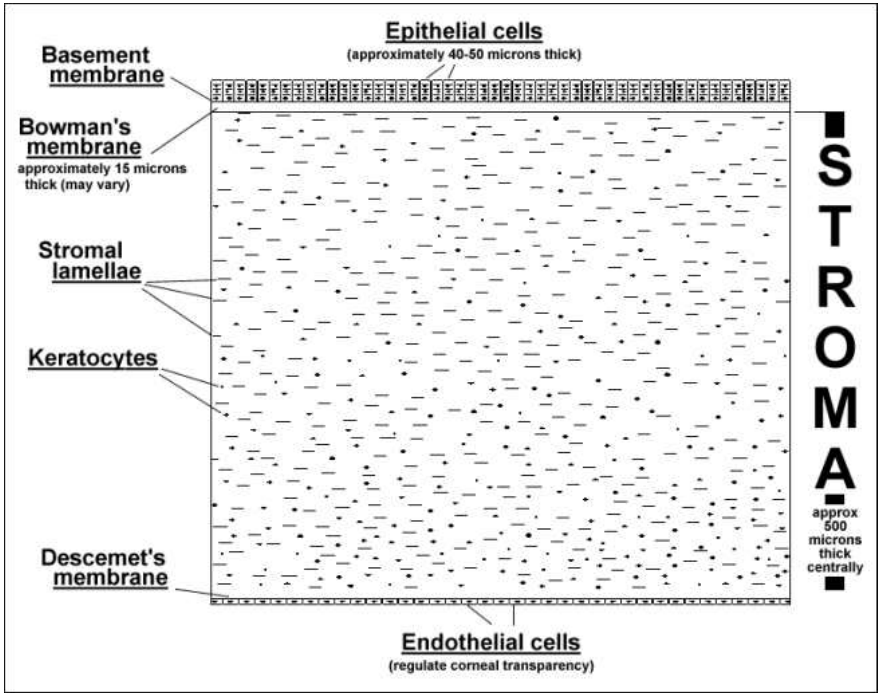
- Predictions from current model deviates from the measurements.
- Many of the parameters are determined indirectly from old models based on “optimal fit”.



Corneal equations



- A two-membrane, three compartment flow model for Cornea



- There will be ***interactions*** between solid, fluid and ionic species

Governing equations

PDEs

$$\begin{cases} \dfrac{\partial W_f}{dt} = -\dfrac{\partial}{\partial \xi}\left(\dfrac{\rho_f}{\rho_s} J_f\right) \\ \dfrac{\partial}{dt}\left(\dfrac{W_f C_k}{\rho_f}\right) = -\dfrac{\partial}{\partial \xi}\left(\dfrac{J_k}{\rho_s}\right) \end{cases}$$

Flux Source

$$J_f = -K_\mu \frac{dP}{dx} = -\frac{dP}{d\xi}\frac{K_\mu}{1 + W_f \rho_s / \rho_f}$$

$$J_k = J_f C_k - \bar{D}_k \frac{\partial C_k}{\partial \xi}\frac{1}{1 + W_f \rho_s / \rho_f}$$

where

$$P = IOP - \gamma \exp(-W_w)$$

$$\frac{W_f}{W_w} = 1 + \frac{\sum_{j=1}^{M} C_j m_j}{C_w m_w} \approx \frac{\rho_f}{\rho_w}$$

Primary Unknowns

W_f - hydration

C_k - ionic concentration

Flux boundary conditions

On the epithelium and endothelium the boundary condition will be flux-type (Natural)

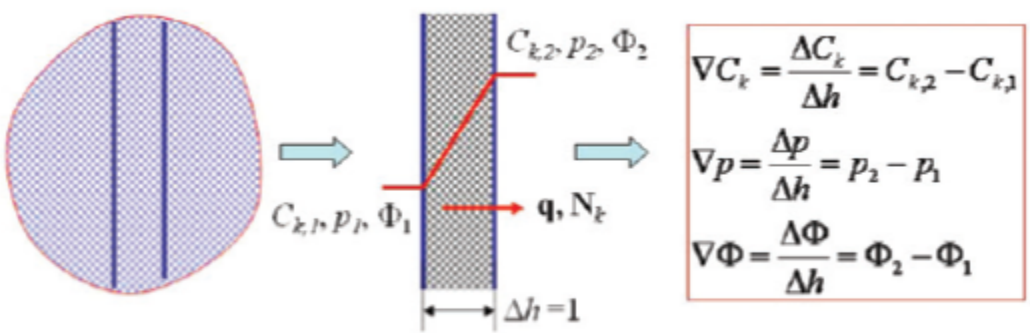
$$(-\bar{n})\cdot\nabla\cdot(-c\nabla\vec{u}-\alpha\vec{u}+\gamma)=\vec{J}=\begin{bmatrix}J_f\\ J_k\end{bmatrix}$$

Where J_f is the volume flux for ionic solution while J_k is the molar flux for species k

$$J_f = L_P\Delta P - L_P\sum_{j=1}^{M}\sigma_j(RT\Delta C_j + z_jC_jF\Delta\phi)$$

$$J_k = (1-\sigma_k)J_fC_k + \omega_k(RT\Delta C_k + z_kC_kF\Delta\phi) + J_{ak}$$

$$\sigma_{Na} \neq \sigma_{Cl} \neq 0$$



Model parameters

Table 1. Parametric values employed in the numerical studies for figures 3–6. (ρ_d=1.49 g cm^{-3} and ξ_{max}=0.007 cm.)

parameter	Na^+	K^+	Cl^-	HCO_3^-
z_k, charge number	1	1	−1	−1
m_k, molar mass (g mole^{-1})	11	19	17	31
C_{kb}, concentration in tears (mole cm^{-3})	130×10^{-6}	15×10^{-6}	110×10^{-6}	35×10^{-6}
L_p, hydraulic conductivity coefficient at epithelium (cm^3 (s dyn)$^{-1}$)		5.8×10^{-12} [a]		
σ_k, reflection coefficient at epithelium	0.800 [a]	0.610 [b]	0.580 [b]	0.800 [b]
$RT\omega_k$, permeability coefficient at epithelium (cm s^{-1})	0.99×10^{-7} [b]	1.45×10^{-7} [b]	1.50×10^{-7} [a]	0.82×10^{-7} [b]
J_k, active pump rate at epithelium (mole (s cm^2)$^{-1}$)	0	0	-1.2×10^{-11} [a,c]	0
P_1, hydrostatic pressures in tears (dyn cm^{-2})			0	
C_{k0}, initial concentration in stroma (mole cm^{-3})	145×10^{-6}	15×10^{-6}	120×10^{-6}	40×10^{-6}
W_{w0}, initial stromal hydration			3.40	
K_μ, flow conductivity coefficient in stroma (cm^2 (s dyn)$^{-1}$)		$8.63\times10^{-15}W_w^4$ [a]		
D_{k0}, diffusion coefficient in stroma (cm^2 s^{-1})		$D_k = D_{k0}\frac{W_f}{W_f+\rho_f/\rho_d}$		
	0.591×10^{-5} [b]	0.867×10^{-5} [b]	0.900×10^{-5} [a]	0.489×10^{-5} [b]
C_{kb}, concentration in aqueous humour (mole cm^{-3})	135×10^{-6}	10×10^{-6}	110×10^{-6}	35×10^{-6}
L_p, hydraulic conductivity coefficient at endothelium (cm^3 (s dyn)$^{-1}$)			53×10^{-12} [a]	
σ_k, reflection coefficient at endothelium	0.685 [b]	0.467 [b]	0.450 [a]	0.828 [b]
$RT\omega_k$, permeability coefficient at endothelium (cm s^{-1})	5.25×10^{-5} [b]	7.71×10^{-5} [b]	8.00×10^{-5} [a]	4.35×10^{-5} [b]
J_k, active pump rate at endothelium (mole (s cm^2)$^{-1}$)	0	0	0	-4.0×10^{-10} [a,d]
P_3, hydrostatic pressures in aqueous humour (dyn cm^{-2})			12.67×10^4 [a]	

Problem Statement

- To find sets of material parameters such that the predicted Δh matches the experiments closely;

- Constraint: the equilibrium state has to be consistent with the physiological condition.

Parameters

$$L_p, \sigma_k, \omega_k$$
$$J_k, K_\mu, D_{k0}$$

PDEs

$$\frac{\partial W_f}{dt} = -\frac{\partial}{\partial \xi}\left(\frac{\rho_f}{\rho_s} J_f\right)$$

$$\frac{\partial}{dt}\left(\frac{W_f C_k}{\rho_f}\right) = -\frac{\partial}{\partial \xi}\left(\frac{J_k}{\rho_s}\right)$$

Osmotic Shock

$$\Delta C(t \in [a,b]) = C_{shock}$$

$$\text{min.} \quad \left|\Delta h(t) - \Delta h^{\exp}(t)\right|, \quad \Delta h(t) = \int_0^{\xi_{\max}} \left(\frac{W_f(t,x)}{\rho_f(t,x)} - \frac{W_{f0}}{\rho_{f0}}\right) \rho_d \, d\xi$$

$$\text{s.t.} \quad \overline{W} = \frac{1}{V} \int_\Omega W_f \, d\Omega \Big|_{t \to \infty} = W_{\text{physiologics}}$$

Questions?

Thank you!