Optimized Stencils for Acute Triangulations

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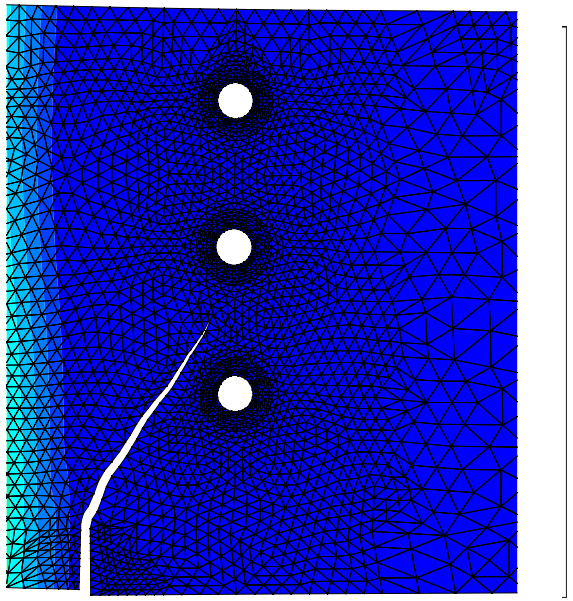
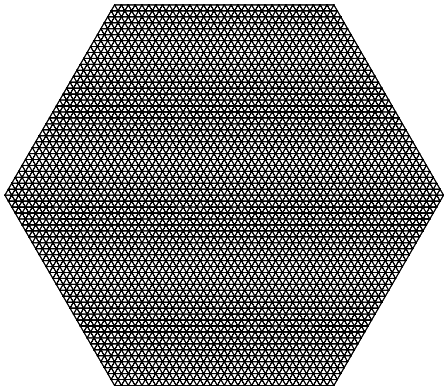
Motivation

- Moving

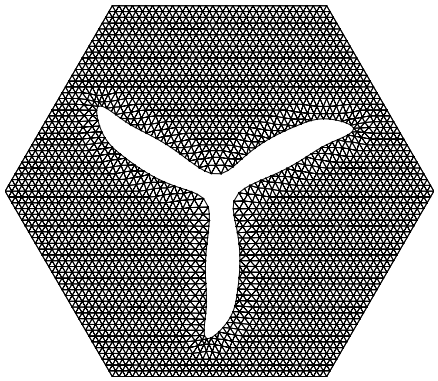
- Fractur



Universal Meshing



Universal Mesh

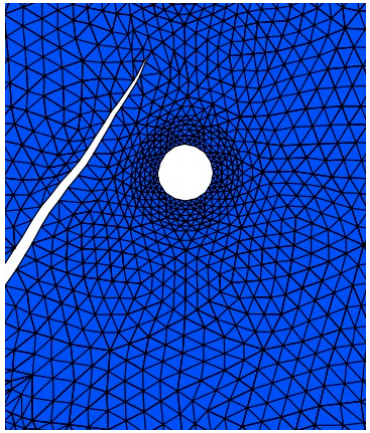


Conforming Mesh

Rangarajan, Lew 2012

Universal Meshing

- Conditions that guarantee it works:
  - The universal mesh is sufficiently refined.
  - The immersed geometry is smooth.
  - All the triangles in the universal mesh should have only acute angles.

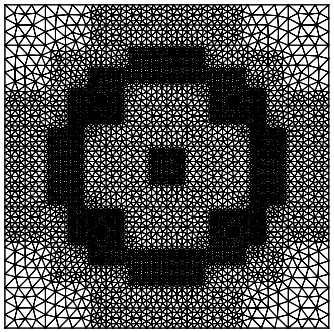


Problem Statement

- Can we build adaptive universal meshes with acute triangulations ?

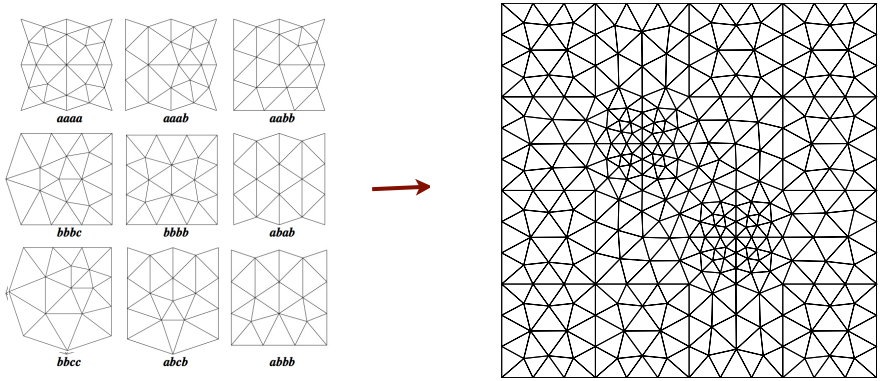
- Build Quadtree based background mesh such that:
  - All the angles in the mesh are acute.
  - Adaptive refinement.

- For example:



Existing Solution

- Bern et al. 1994 gave the stencils with acute triangulations for quadtree based mesh generatior



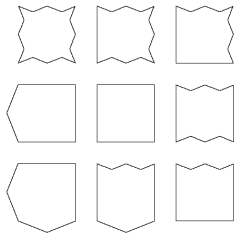
- Angle range 36^0 - 80^0

Proposed Methodology

In order to build similar stencils , we divide this problem into 3 steps:

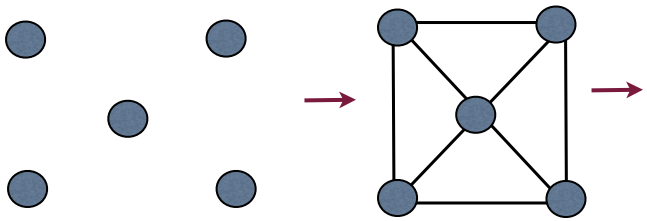
- Nodal location optimization
- Triangulation
- Angle optimization

- We fix the faces of the stencils.



Nodal location optimization

- For example: the case of bbbb



Best angle possible is 90^0!

Number of interior points play important role.

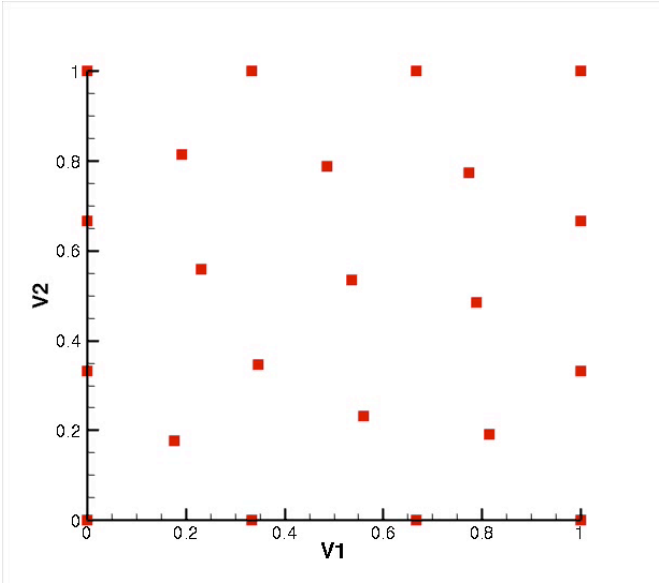
- We 'throw' in the nodes and optimize potential function V.

$$V = \sum_{\forall(pt_1, pt_2)} \frac{1}{C\,p} X^{-p}$$

Where, $X = \text{dist}(pt_1, pt_2)$

C is constant and p=4

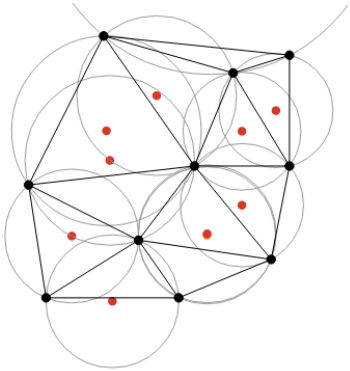
Nodal location optimization



- Optimization process for bbbb stencil with 10 interior points

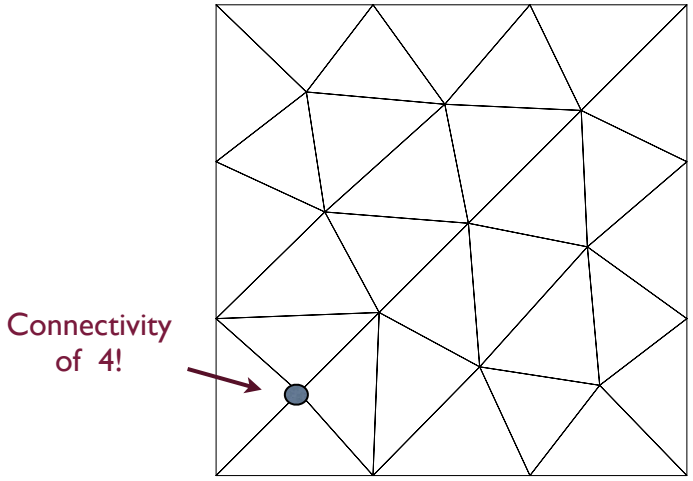
Delaunay Triangulation

- " In mathematics and computational geometry, a Delaunay triangulation for a set P of points in a plane is a triangulation DT (P) such that no point in P is inside the circumcircle of any triangle in DT(P). Delaunay triangulations maximize the minimum angle of all the angles of the triangles in the triangulation; they tend to avoid skinny triangles."



Triangulation

We use Delaunay triangulation to generate mesh over the fixed location of nodes for each stencils.



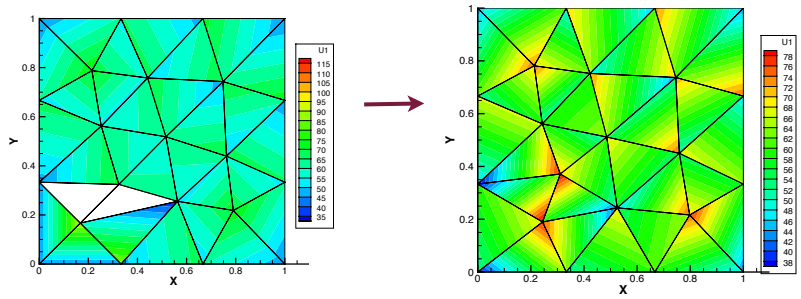
Angle Optimization

We optimize the potential function E to achieve acute triangulation.

$$E = \sum_{\theta \in M} (\cos\theta - \frac{1}{2})^2$$

Edge Flipping changes connectivity so that:

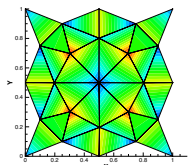
- Interior nodes have connectivity of 5 or more.
- Boundary nodes have connectivity of 4 or more.

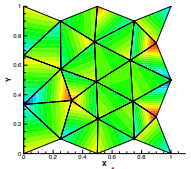


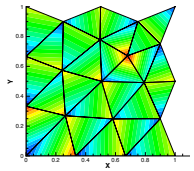
The contours represent the angles, triangles with an obtuse angle are blanked

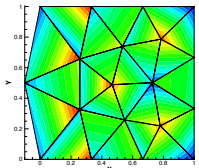
Optimized Stencils

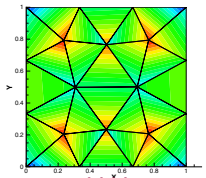
- Angle range 41^0-78^0

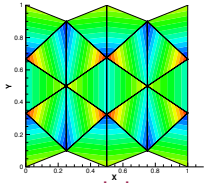

aaaa

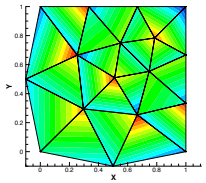

aaab

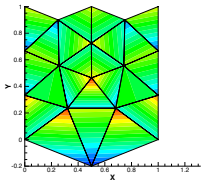

aabb

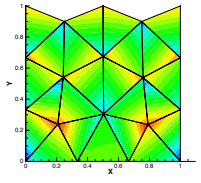

bbbc


bbbb


abab


bbcc


abcb

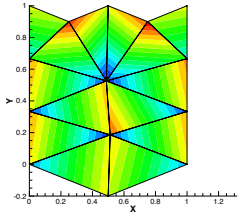
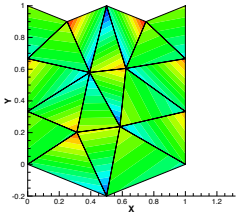
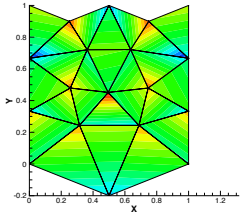

abbb

The contours represent the angles

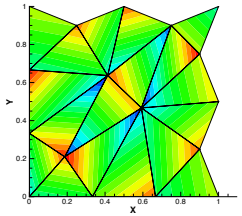
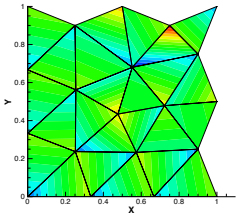
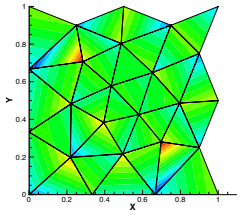
Family of Stencils

- We achieve family stencils for the same face combinations.

For example abcb stencil

aabb stencil

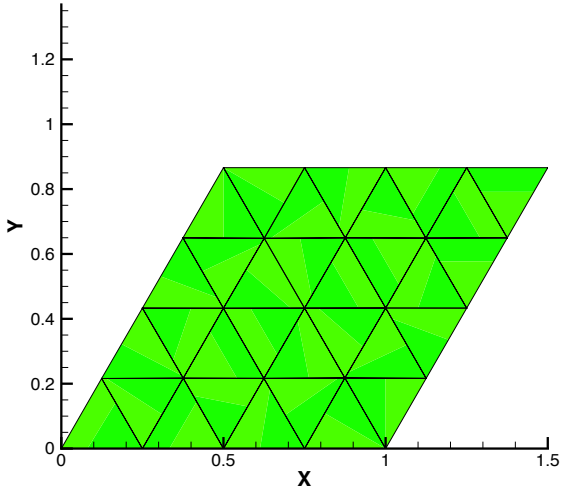
The contours represent the angles

Family of Stencils

Stencil	Number of interior points
aaaa	4, 5, 6, 7, 8, **9**, 10, 11, 12, 13, 14
aaab	4, 5, 6, **7**, 8, 9, 10, 11, 12
aabb	3, 5, 6, **7**, 8, 9, 10
abab	1, 3, **4**, 5, 6, 7, 8, 9, 10
abbb	**6**, 7, 8, 9, 10, 11, 12
bbbb	**8**, 9, 10, 11, 12, 13, 14
abcb	2, 3, 4, **5**, 6, 7, 8, 9, 10, 11
bbcc	**7**, 8, 9, 10, 11, 12
bbbc	**8**, 9, 10, 11, 12, 13

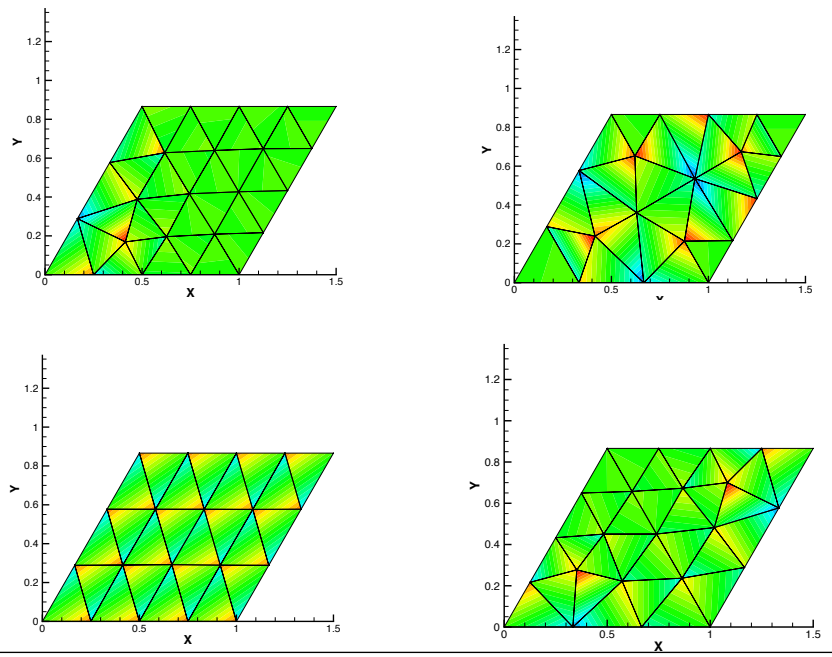
Any better stencils ?!

Diamond Stencils

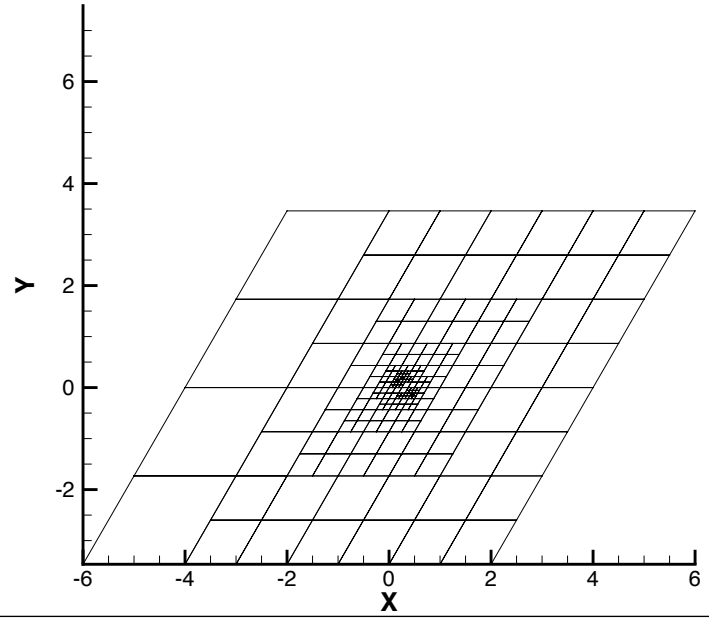


Instead of square, use rhombus to get better angles.
All angles 60!

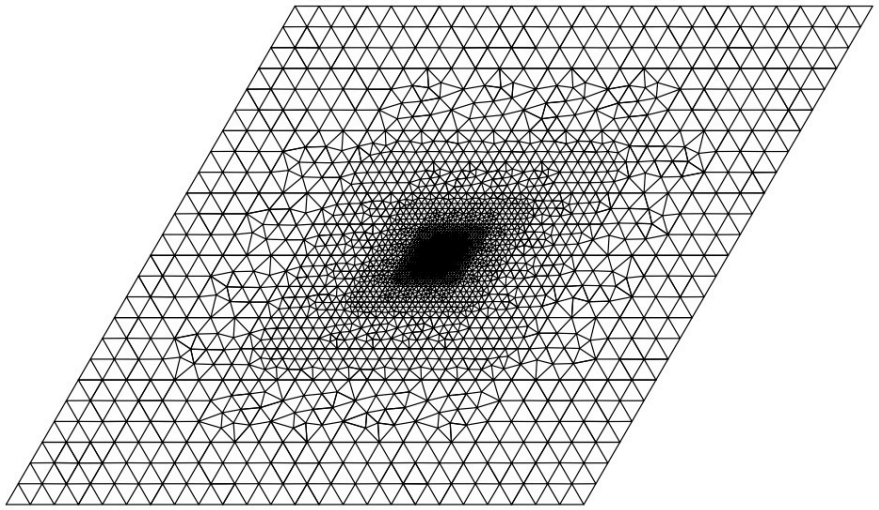
Diamond Stencils



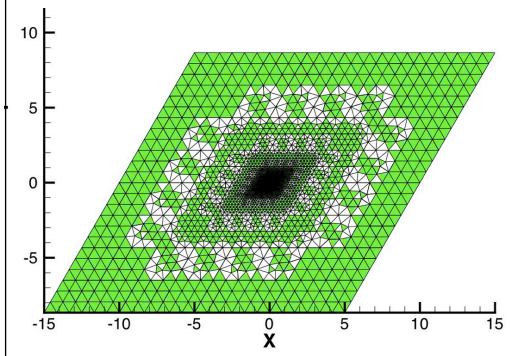
Diamond - Tree



Mesh from Diamond - Tree

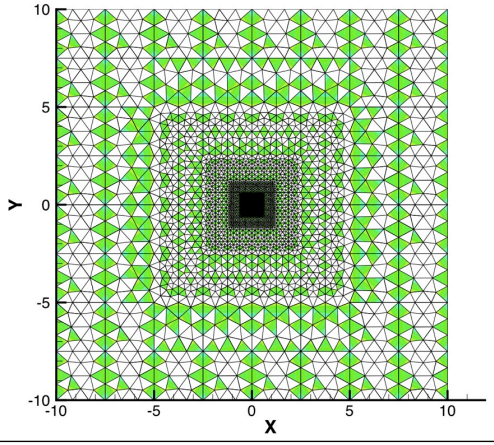


Comparison

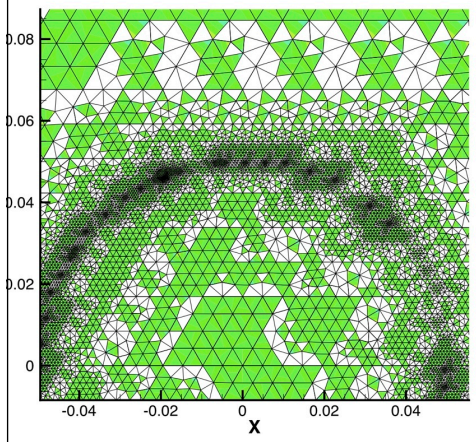


Diamond stencils based

Square stencils based

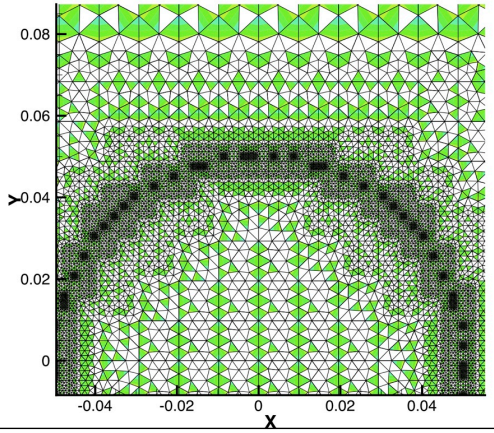


Comparison



Diamond stencils based

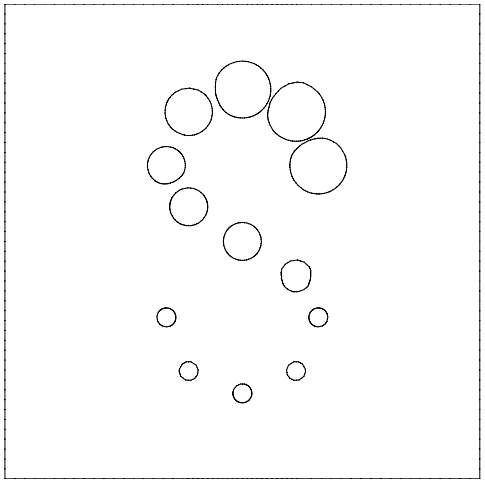
Square stencils based




Application

Numerical Examples

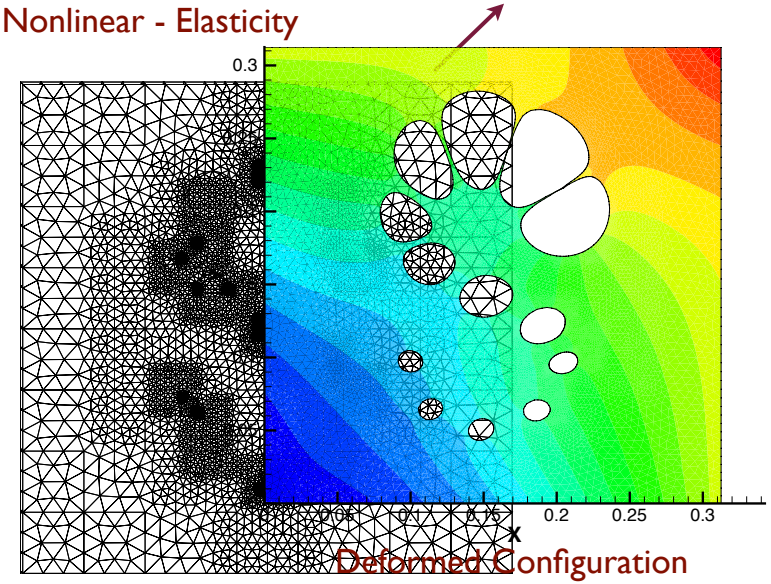
- Distorting 'S' : Nonlinear - Elasticity



Domain of analysis

Numerical Examples

- Distorting 'S' : Nonlinear - Elasticity



Deformed Configuration

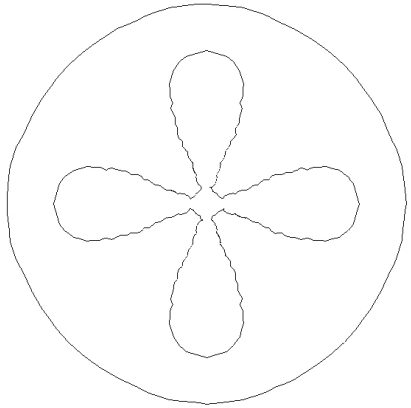
Universal mesh

Conforming mesh

The contours represent the net displacement

Numerical Examples

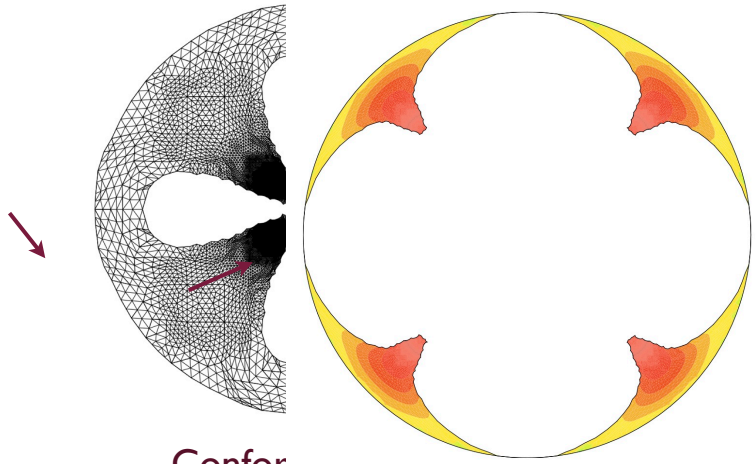
- Expanding Cavity : Large deformation



Domain of analysis

Numerical Examples

- Expanding Cavity : Large deformation



Confor

Universal Mesh

Deformed Configuration

Concluding Remarks

- We have set up an optimization problem leading us to stencils having all acute angles.

- Through this setup we improve the stencil provided in Bern et al. 1994.

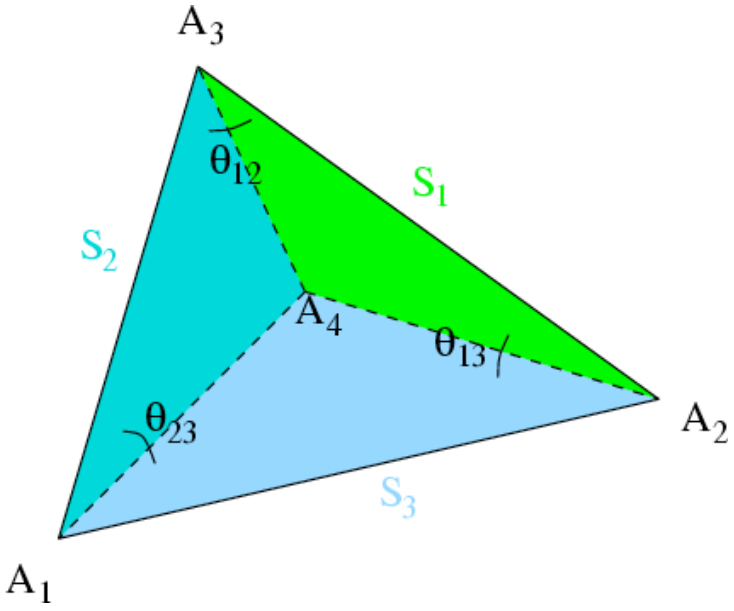
- This enables us to achieve family stencils for the same face combinations by varying number of interior points.

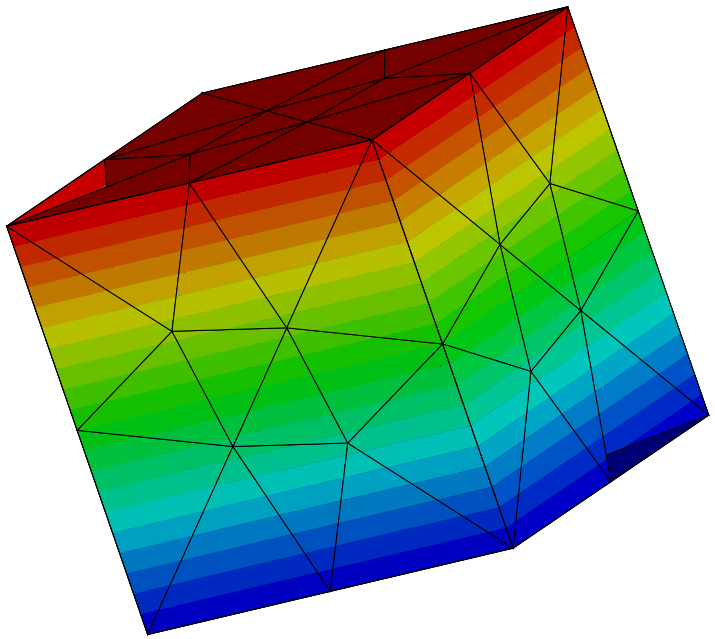
- This leads to better acute angle universal mesh.

- This optimization problem would serve as primary step to achieve acute tetrahedralization of a cube.

Tetrahedralize 3D Domain!

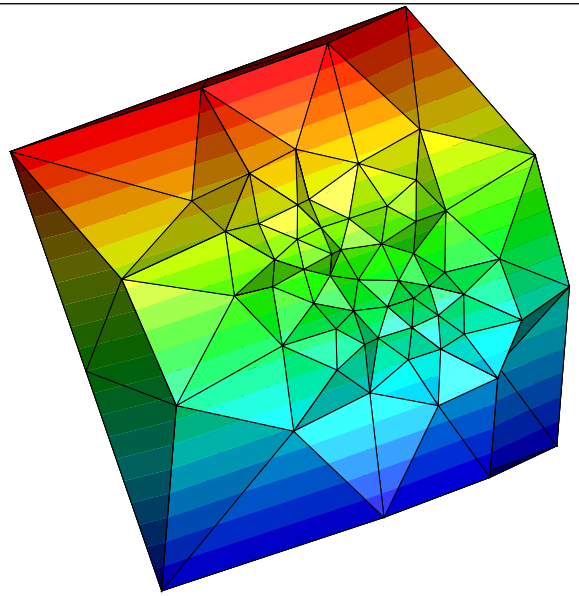
Acute Tetrahedron





1370 Tets, All actue dihedral angles

Vanderzee. E, et al 2012

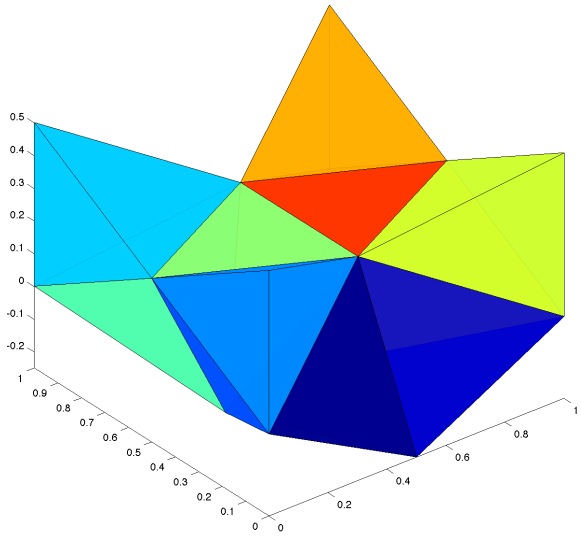


Size of the tetrahedron vary a lot, leads to bad mesh.

Vanderzee. E, et al 2012

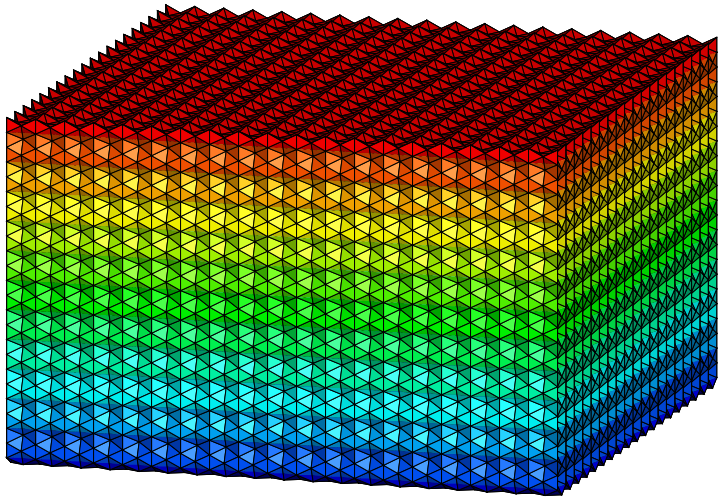
Next Idea ?
Tetrahedral Close Packing

Building Block



- Place nodes similar to BCC lattice.
- Apply off the shelf Delaunay Tetrahedralization.
- Apply Constraint to form a tetrahedron only when acute condition is satisfied

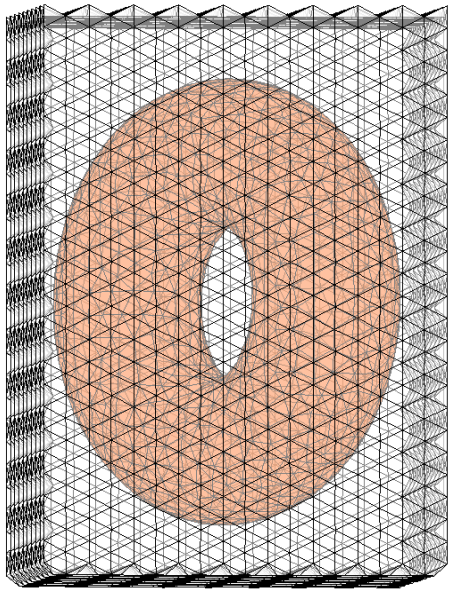
Repeat in 3D: “Universal Mesh”

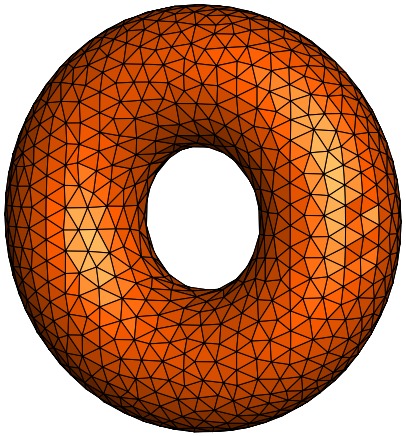


Size of the tetrahedrons do not vary, all dihedral angles less than 77 degree.

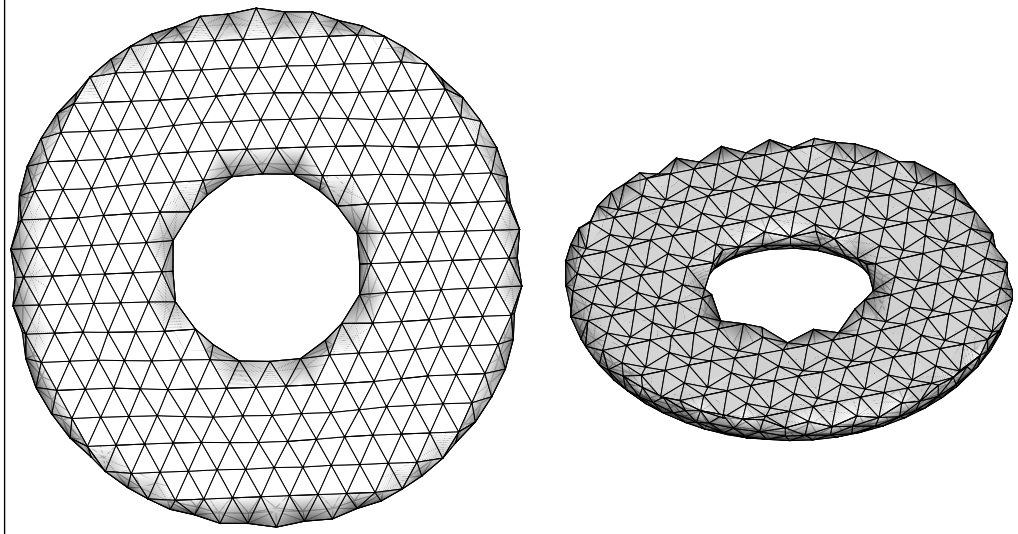
Test on 3D geometries

Torus

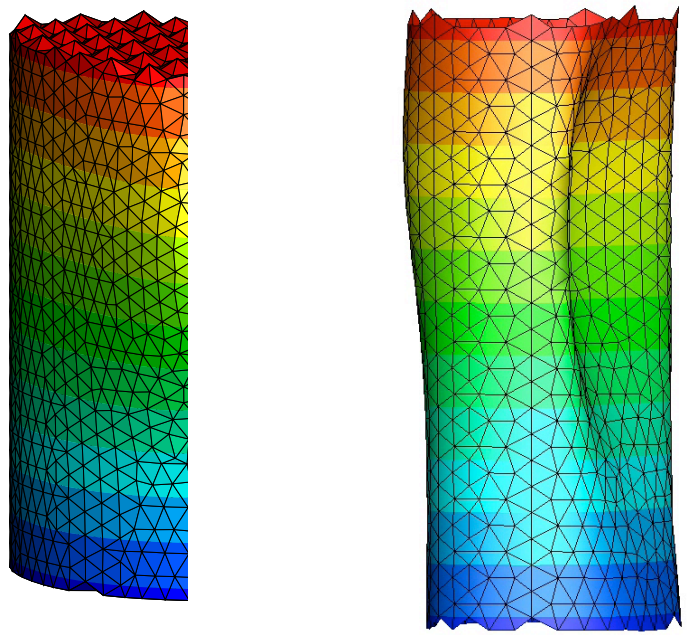




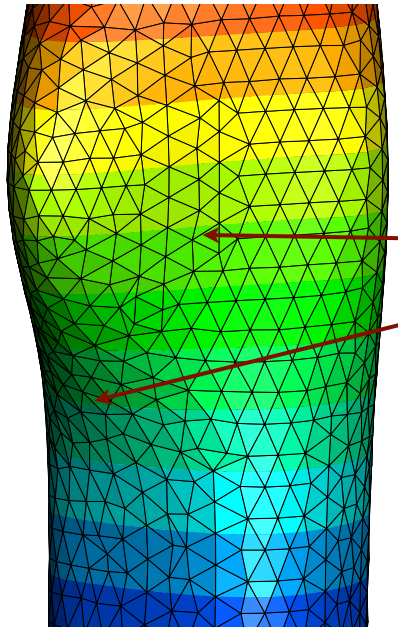
Torus



Changing Geometry



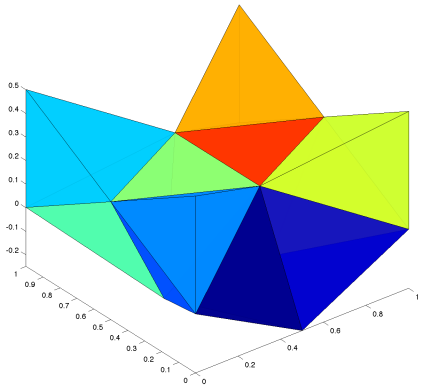
But is it efficient ?

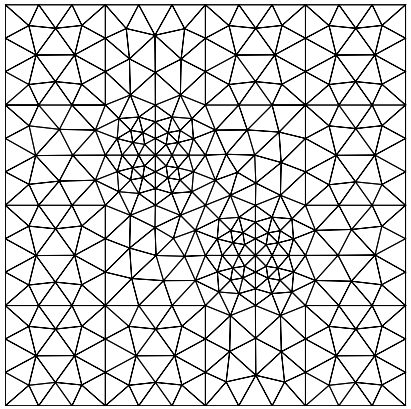


Size of the tetrahedrons doesn't have to be the same.

As one can establish bound on the minimum size of face of the tetrahedron in terms of the local curvature.

Transition in the size..





19 stencils required for Oct-tree. Build Stencils based on building block similar to those done in 2D.