Improving the robustness of Newton-based power flow methods to cope with poor initial points

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Abstract

Solving power flow problems is essential for the reliable and efficient operation of a power network. However, current software for solving these problems have questionable robustness due to the limitations of the solution methods used, which are typically based on the Newton-Raphson method. One limitation is that a "good" initial point is usually required to obtain a solution. We explore homotopy-based techniques to mitigate this limitation. These techniques have been tested on large power flow test cases with poor initial points and it can be seen that they can lead to methods that are much more robust than traditional methods.

1 Introduction

The power flow problem consists of determining the steady-state operating point of an electrical power network. That is, to determine the voltage magnitudes and angles of every bus as well as the active and reactive power flows through every branch [1] [2]. Solving power flow problems is essential for the reliable and efficient operation of a power network, as many analyzes and processes depend on the information provided by the solutions. Expansion planning, voltage stability analysis, outage scheduling and contingency analysis, for example, all depend on the availability of power flow solvers. Unfortunately, current software used for solving these problems have questionable robustness. This lack is primarily an algorithmic issue and may not reflect physical problems of the systems being analyzed [5]. Failure to solve power flow problems can leave system operators or planners with the difficult task of trying to get a case to converge, which may require running multiple methods or tweaking the networks, or with limited or incorrect knowledge about the system.

This lack of robustness comes from the solution method used, which typically consists of the Newton-Raphson (NR) method for solving systems of nonlinear equations combined with heuristics for enforcing reactive power limits. This method is simple to implement and can exhibit a quadratic rate of convergence, but it suffers from serious robustness issues. Its convergence, for example, is only local. That is, a point sufficiently close to the solution must be given, otherwise, the method can diverge or cycle [10]. Unfortunately it is rarely known whether the initial point is sufficiently close. The heuristics, often called "PV-PQ switching", can also affect convergence by causing jumps or cycles in the numerical process [13].

Many techniques have been proposed in the literature for improving the robustness of this method. Among them are the use of an "optimal multiplier" [6] and the use of integration or gradient flow techniques [9] for improving convergence. These techniques, although very helpful for solving more ill-conditioned problems, do not resolve the issue of requiring a good initial point for solving power flow problems. In this work we describe the development of techniques for overcoming this limitation. Specifically, we describe techniques based on homotopy and provide results showing their performance on large-scale power flow cases.

2 Problem formulation

Mathematically, the power flow problem for a network of $n \in \mathbb{N}$ buses consists of solving the nonlinear system of equations [1] [2]

$$\Delta P_k = P_k - \sum_{m \in [n]} v_k v_m (G_{km} \cos(\theta_k - \theta_m) + B_{km} \sin(\theta_k - \theta_m)) = 0, \quad k \in [n] \tag{1}$$

$$\Delta Q_k = Q_k - \sum_{m \in [n]} v_k v_m (G_{km} \sin(\theta_k - \theta_m) - B_{km} \cos(\theta_k - \theta_m)) = 0, \quad k \in [n], \tag{2}$$

where $P + jQ \in \mathbb{C}^n$ are the net injected powers (generation minus load), $G + jB \in \mathbb{C}^{n \times n}$ is the constant network admittance matrix, $v \in \mathbb{R}^n_+$ are the bus voltage magnitudes, $\theta \in \mathbb{R}^n$ are the bus voltage angles and $[n] = \{1, \dots, n\}$.

Typically, one of the generator buses, say $s \in [n]$, is chosen as a slack bus. The function of this bus is to provide sufficient power to account for any load or losses not known in advance as well as to provide a reference angle from which all other angles are measured [1] [2]. The rest of the buses are partitioned into the set $\mathcal{R}$ of (voltage) regulated buses and the set $\mathcal{U}$ of unregulated buses. Regulated buses have generators attached to them and these generators have the capability of adjusting their reactive power to maintain their bus voltage magnitude fixed (regulated) at some target.

An important property of the problem is that each bus of the network has exactly two unknowns. Table 1 shows what these unknowns are for each of the different bus types. Since each bus has two equations associated to it, *i.e.*, active and reactive power balance, the system of equations that must be solved has an equal number of variables and constraints [1] [2].

Bus	Type	Variable	Fixed
$k = s$	slack	P_k and Q_k	v_k and θ_k
$k \in \mathcal{U}$	unregulated	v_k and θ_k	P_k and Q_k
$k \in \mathcal{R}$	regulated	Q_k and θ_k	P_k and v_k

Table 1: Variable and fixed quantities

The voltage magnitudes of the regulated buses are not always fixed at their targets. When the reactive power of a generator providing regulation hits its limit, its bus voltage magnitude is allowed to move away from its target, effectively turning the bus into an unregulated bus. However, we assume that regulating generators do not have reactive power limits and hence do not include this phenomenon in our analysis. The reason for doing this is that reactive power limits of regulating generators are not directly related to the robustness of Newton-based methods against poor initial points. This phenomenon will be addressed in future work where it has a greater relevance.

3 Literature review

Several ideas have been proposed in the literature for reducing the performance dependency of power flow methods based on Newton-Raphson on having a good initial point. In [12], for example, the author proposes first obtaining the bus voltage angles from a DC power flow, computing the bus voltage magnitudes from the angles in a direct way using approximations, and then using these values as a starting point for the Newton-Raphson method. They show that such an initial point is a better initial point than the flat starting point for several test cases, the largest of which has 205 buses. A similar idea is proposed in [8]. This author also proposes using the solution of a linear power flow problem as an initial point and compares its performance against using a flat starting point on a few IEEE test cases of up to 57 buses. A more elaborate idea based on homotopy, the technique we also explore in this research, is proposed in [3]. These authors propose solving first the DC power flow to obtain an initial point and then *gradually* transforming this problem into the AC power flow. They test this idea on several IEEE test cases and on two ill-conditioned problems, the largest of which has 2383 buses, and compare the number of function evaluations required by this method with those required by MATLAB's `fsolve` routine, but provide no robustness results.

4 Benchmark methods

To evaluate the techniques explored in our research, we compare their performance against that of the most widely used method for solving power flow problems and an improved version of it. These benchmark methods are described in the following subsections.

4.1 Newton-Raphson (NR)

The NR method is a well known iterative algorithm for solving systems of nonlinear equations of the form $f(x) = 0$, where f is continuously differentiable. The method updates an approximate solution x_k at iteration k by requiring that the linearized system of equations $f_k + J_k(x_{k+1} - x_k) = 0$ is satisfied, where J_k is the Jacobian matrix of f at x_k and $f_k = f(x_k)$. This results in the new approximate solution being $x_{k+1} = x_k + p_k$, where p_k satisfies the "Newton system" $J_k p_k = -f_k$. Typically, this process is terminated when x_k satisfies $\|f(x_k)\|_\infty < \epsilon$ for some predefined $\epsilon > 0$ or when a maximum number of iterations is reached. As explained in [10], this method can exhibit a quadratic rate of convergence but it is only locally convergent.

In the context of power flow problems, the function f is the vector-valued function of "power mismatches", which gives $\{\Delta Q_i\}_{i \in \mathcal{U}}$ and $\{\Delta P_i\}_{i \in [n] \setminus \{s\}}$. The variable x consists of the voltage magnitudes $\{v_i\}_{i \in \mathcal{U}}$ and the angles $\{\theta_i\}_{i \in [n] \setminus \{s\}}$.

4.2 Line Search Newton-Raphson (LSNR)

The NR method can be augmented with a line search procedure to improve its range of convergence. A line search procedure is a common procedure used inside optimization algorithms for choosing the step length α_k along a particular direction p_k

from the current iterate x_k to the next, *i.e.*, to obtain $x_{k+1} = x_k + \alpha_k p_k$, and for trying to ensure progress towards a solution. The function reduced in the line search is the univariate function $\|f(x_k + \alpha p_k)\|_2^2$. Consequently, $\|f(x_{k+1})\|_2 < \|f(x_k)\|_2$. Without a line search, NR can not ensure iterates improve in this or any other sense unless the initial point is close to the solution. How "close" the initial point needs to be is not generally known.

For this study, we use a line search procedure based on the *strong Wolfe conditions* [11] and our implementation uses a simple bracketing and bisection strategy [4].

5 Homotopy-based power flow methods

First we describe the homotopy approach in general terms and then describe the three homotopy-based power flow methods explored in our research and present results comparing their performance for initial points of varying quality.

5.1 General approach

The homotopy approach consists of replacing the original problem with a sequence of subproblems that have certain properties: the first subproblem in the sequence is easy to solve, the last or limiting subproblem is equivalent to the original problem, and consecutive subproblems in the sequence are closely related [7] [10]. Although eventually the original problem needs to be solved by then the hope is that the initial point used is close to the solution.

More formally and in the context of solving the power flow equations $f(x) = 0$, the homotopy approach consists of solving a sequence of subproblems $h(x, t_k) = 0$, $k \in \mathbb{N}$, where $t_1 = 0$, $t_k \to 1$ as $k \to \infty$, $h(x, 0) = 0$ is easy to solve from the available initial point, which we denote by x_0, $h(x, 1) = 0 \iff f(x) = 0$, and $h(x, t_k) \approx h(x, t_{k+1})$ in the sense that the solution of $h(x, t_k) = 0$ is a good initial point for solving $h(x, t_{k+1}) = 0$.

5.2 Injection homotopy (IH)

A simple and common way to apply the homotopy approach to the system of nonlinear equations $f(x) = 0$ is to use

$$h(x, t) = f(x) - (1 - t)f(x_0), \tag{3}$$

where x_0 is the available initial point [10]. As can be easily seen, for $t = 0$, the system $h(x, 0) = f(x) - f(x_0) = 0$ is trivially solved by setting $x = x_0$, and for $t = 1$, $h(x, 1) = f(x)$ so the solution of $h(x, 1) = 0$ is the solution of the power flow problem. Clearly, provided the Jacobian of f is not singular at x_0, there exists $t \in (0, 1)$ such that x_0 is sufficiently close for NR to converge.

An interpretation of this approach is that first, generators and loads (positive and negative injections, respectively) are added to the network in such way that x_0 solves the power flow equations. Then, they are gradually removed until the network returns to its original state. The name "injection homotopy" reflects this interpretation.

5.3 Phase homotopy (PH)

Another way to apply the homotopy approach for solving the power flow problem is to let h be a function such that $h(x, t) = 0$ represents the power flow equations for a transformed network that has line phase shifts given by

$$\psi_{km}(t) = (1 - t)\phi_{km}^0 + t\phi_{km}, \tag{4}$$

where m and k denote adjacent buses, ϕ_{km} denote the original line phase shifts and ϕ_{km}^0 denote new phase shifts, which are chosen in such way that x_0 closely approximates the system's state, *i.e.*, $h(x_0, 0) \approx 0$. The new line phase shifts can be obtained by approximately minimizing $\|h(x_0, 0)\|_2^2$ over line phase shifts, and thus extra computations are required.

An interpretation of this approach is that first, line phase shifts are added to the network in such way that x_0 approximately solves the power flow equation. Then, they are gradually removed until the network is in its original state. The name "phase homotopy" reflects this interpretation.

5.4 Performance of IH and PH

Both the IH method and the PH method try to solve the power flow problem by solving a sequence of subproblems (approximately) of the form

$$\underset{x}{\text{minimize}} \quad \frac{1}{2}\|h(x, t_k)\|_2^2, \tag{5}$$

where $t_1 = 0$ and $t_k \to 1$ as $k \to \infty$. In our implementation, we used $t_k = \min\{1, t_{k-1} + \Delta t\}$ for $k > 1$, where Δt is a constant, say 0.2, and used the LSNR method to solve each subproblem.

To test these methods, we performed the following experiment: for each of the collected power flow test cases and for increasing values of standard deviation, we constructed 30 distinct initial points by adding zero-mean Gaussian perturbations

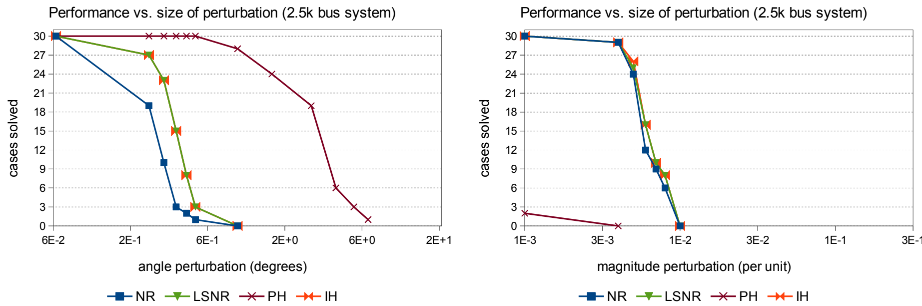

Figure 1: Performance of the NR, LSNR, PH and IH methods on the 2.5k-bus test case.

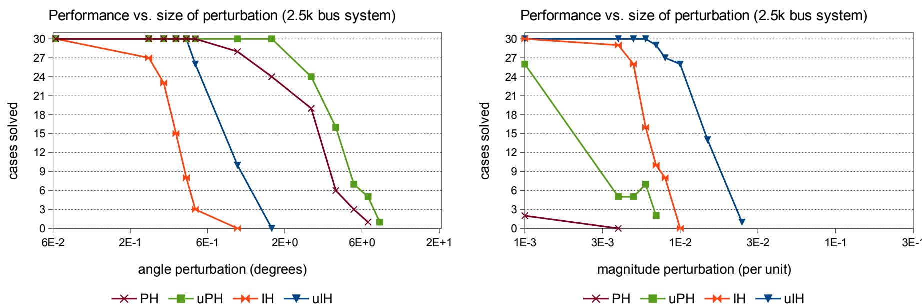

Figure 2: Effects of including preferences.

to the given good initial point available from the data. We then ran the methods described above with the constructed initial points and recorded how many of the 30 cases were solved as a function of the size of the perturbations (standard deviation). We considered a case solved if the error in the power flow equations was reduced below a tolerance (10^{-4} per unit system MVA) and the minimum voltage magnitude over all buses at the solution was greater than 0.5 per unit nominal voltage. The test cases we collected came from a 1.2k-bus European network, and 2.5k-bus and 46k-bus networks of two distinct North American Independent System Operators (ISO). Figure 1 shows the results obtained for the 2.5k-bus system. The diagram on the left corresponds to voltage angle perturbations and the one on the right to voltage magnitude perturbations.

As can be seen in Figure 1, the IH method did not outperform the LSNR method while the PH method was able to solve many more cases than the other methods for larger voltage angle perturbations, but performed the worst for voltage magnitude perturbations. Similar results were obtained for the 1.2k-bus and 46k-bus systems with the exception that for the 46k-bus system, the PH method also performed the worst for voltage angle perturbations.

5.5 Including preferences

By analyzing the results obtained from the perturbation experiments described above and observing how the IH and PH methods failed, we concluded that for poor initial points, these methods may be attracted to regions of points having near zero voltage magnitudes at some buses and possibly very large voltage magnitudes at others, even though physically meaningful solutions were known to exist. To address this issue, we included preferences for points having near one voltage magnitudes in each of the subproblems (5), which have the impact of making the region of physically meaningful solutions attractive. More specifically, instead of solving the sequence of subproblems (5), we solved a sequence of subproblems of the form

$$\underset{x}{\text{minimize}} \quad u_k \varphi(x) + \frac{1}{2}\|h(x,t_k)\|_2^2, \tag{6}$$

where $t_1 = 0$, $t_k \to 1$ and $u_k \to 0$ as $k \to \infty$, and

$$\varphi(x) = \frac{1}{2}\sum_{i \in \mathcal{U}} (v_i - 1)^2. \tag{7}$$

Figure 2 shows the effects of including preferences to avoid searching for solutions in regions with inappropriate bus voltage magnitudes. The names uIH and uPH correspond to the IH and PH methods, respectively, with the preferences included. As the figure shows, the inclusion of preferences (7) significantly improves the robustness of the IH and PH methods on cases with poor initial points.

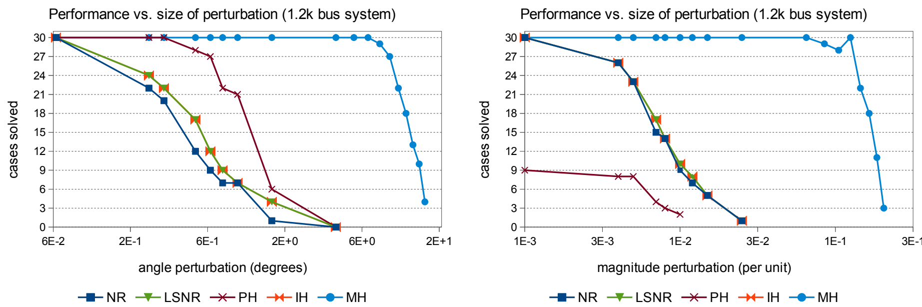

Figure 3: Performance of all the methods on the 1.2k-bus test case.

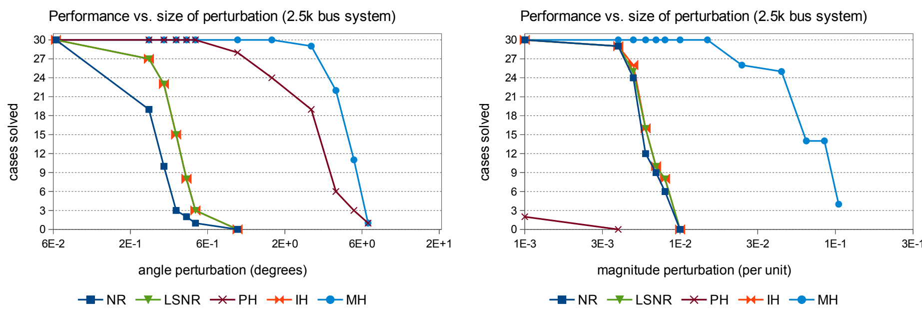

Figure 4: Performance of all the methods on the 2.5k-bus test case.

5.6 Magnitude homotopy (MH)

Including preferences to encourage the solution method to search for solutions in "good" regions helps not only obtain more meaningful solutions, but also to solve more cases, as we saw in Figure 2. This idea, in fact, may be all that is needed for achieving robustness against poor initial points. To test this conjecture, we developed a third homotopy-based method, which we call "magnitude homotopy". This method applies a homotopy approach based only on the preferences stated above and does not modify the network, *i.e.*, it is noninvasive. More specifically, the MH method solves the power flow problem by solving a sequence of subproblems (approximately) of the form

$$\underset{x}{\text{minimize}} \quad (1-t_k)\varphi(x) + t_k \frac{1}{2}\|f(x)\|_2^2, \tag{8}$$

where $t_1 = 0$ and $t_k \to 1$ as $k \to \infty$. In our implementation, we used $t_k = \min\{1, t_{k-1} + \Delta t\}$ for $k > 1$, where Δt is a constant, say 0.2, and used the LSNR method to solve each subproblem.

This method can be interpreted as the homotopy approach of Section 5.1 with

$$h(x,t) = (1-t)\nabla\varphi(x) + tJ(x)^T f(x), \tag{9}$$

where $J = \frac{df}{dx}$ or as the *penalty function method* [4] applied to the optimization problem

$$\underset{x}{\text{minimize}} \quad \varphi(x) = \frac{1}{2}\sum_{i\in\mathcal{U}}(v_i - 1)^2 \tag{10}$$

$$\text{subject to} \quad f(x) = 0. \tag{11}$$

5.7 Performance of MH

To test the performance of the MH method, we repeated the perturbation experiment described in Section 5.4. Figures 3, 4 and 5 show the performance of all the methods for the 1.2k-bus, 2.5k-bus and 46k-bus test cases, respectively. As the figure shows, the MH method outperformed all the other methods by solving many more cases for larger perturbations of the initial point.

6 Conclusions

In this research, we focused on increasing the robustness of NR-based methods against poor initial points. To do this we first applied a homotopy approach in two different ways: by modifying injections and by modifying line phase shifts.

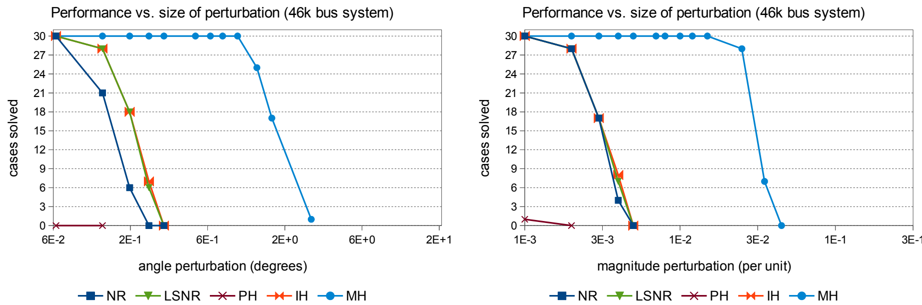

Figure 5: Performance of all the methods on the 46k-bus test case.

The methods based on these ideas (IH and PH) did not provide satisfactory robustness results. The key issue was that nothing prevented these methods from generating very large or near zero bus voltage magnitudes during the solution process, preventing convergence or resulting in meaningless solutions. To fix this issue we experimented with adding preferences in the formulation of the problem to help guide the search for a solution towards points with meaningful bus voltage magnitudes. This led to the development of a third method (MH) that combined preferences and the homotopy approach for solving power flow problems. The robustness of this method and its superiority over the other methods was demonstrated by testing all the methods on large power flow test cases with perturbed initial points. We are currently extending the MH method to control voltage magnitude regulation and handle reactive power limits and stability constraints using optimization techniques, in order to avoid the common, and often problematic, "PV-PQ" switching heuristics.

7 Acknowledgment

Support for this work was provided by the Electric Power Research Institute and the Office of Naval Research (Grant: N00014-11-1-0067).

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