Optimal Power Flow

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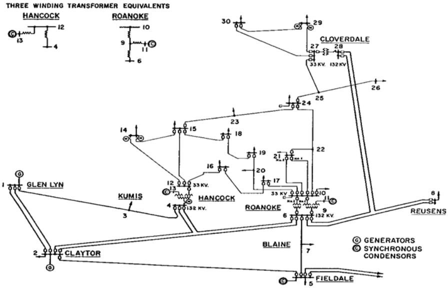
The problem



Configurate generators to

minimize	total generation cost
subject to	supply meets demand
	power flow constraints
	various capacity constraints

"Optimal Power Flow (OPF)"

The problem

Importance: OPF is solve routinely to decide

- how much power to generate where (resource allocation)
- pricing and parameter configuration (e.g. transformer tapping ratio)

Difficulty:

- nonconvex
- practice use heuristic feasible solution or local minimizer

The problem

- Power network: $\mathcal{G}(\mathcal{V}, \mathcal{E})$
- Complex voltage: $v \in \mathbb{C}^n$
- Complex power: $p + iq \in \mathbb{C}^n$
- Complex current (nodal injection): $Yv \in \mathbb{C}^n$, where $Y \in \mathbb{C}^{n\times n}$ defined by

$$Y_{jk} = \begin{cases} \sum_{l\sim j} y_{jl} + y_{jj} & j = k \\ -y_{jk} & j \sim k \\ 0 & j \not\sim k \end{cases}$$

- Cost: $\sum_{j\in\mathcal{V}} C_j(p_j)$ quadratic, convex, increasing

Find v, p, q that

$$\begin{array}{lll} \text{minimize} & \sum_{j\in\mathcal{V}} C_j(p_j) & \\ \text{subject to} & \underline{v}_j \le |v_j| \le \overline{v}_j, & j \in \mathcal{V} \\ & \underline{p}_j \le p_j \le \overline{p}_j, & j \in \mathcal{V} \\ & \underline{q}_j \le q_j \le \overline{q}_j, & j \in \mathcal{V} \\ & p_j + iq_j = v_j(Yv)^*_j, & j \in \mathcal{V} \end{array}$$

- quadratic constrained quadratic program (e.g. eliminate p and q)
- nonconvex constraints

omit line capacity constraints for now

Semidefinite program reformulation

- Eliminate p, q

$$p_j = v^* A_j v,$$

$$q_j = v^* B_j v,$$

where $A_j = \frac{1}{2}(e_j e_j^* Y + Y^* e_j e_j^*)$, $B_j = \frac{1}{2i}(Y^* e_j e_j^* - e_j e_j^* Y)$.

- Linear cost for the moment

$$\sum_{j \in \mathcal{V}} c_j p_j = tr(Mvv^*),$$

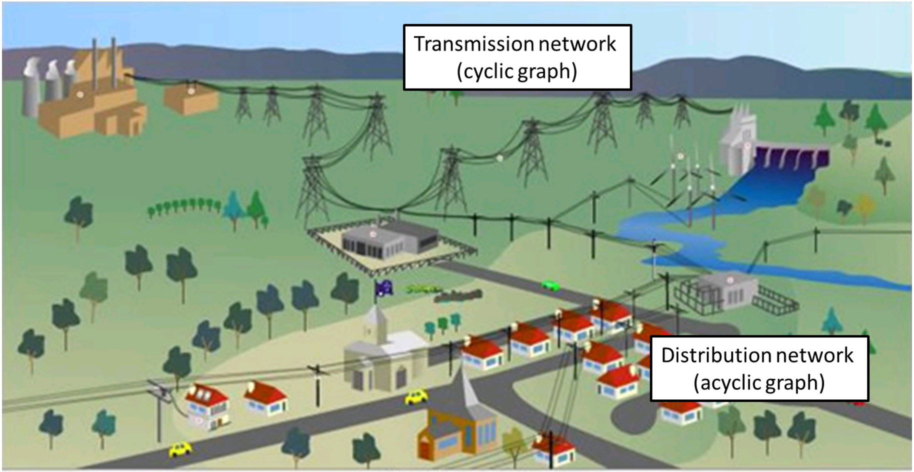
where $M = \frac{1}{2}(CY + Y^* C)$, $C = \mathbf{diag}(c_1, \ldots, c_n)$

Let $W = vv^*$, solving for W with rank 1 constraint

$$\begin{array}{lll} \text{minimize} & tr(MW) & \\ \text{subject to} & \underline{v}_j \le W_{jj} \le \overline{v}_j, & j \in \mathcal{V} \\ & \underline{p}_j \le tr(A_j W) \le \overline{p}_j, & j \in \mathcal{V} \\ & \underline{q}_j \le tr(B_j W) \le \overline{q}_j, & j \in \mathcal{V} \\ & rank(W) = 1 & \end{array}$$

- Removing the rank one constraint obtains a convex problem (SDP relaxation)
- If SDP relaxation finds a solution s.t. optimal W has rank one, the original problem is solved

use a trick based on Schur complement, quadratic cost problem has similar SDP reformulation [LL12]



Electricity network

Distribution network: $\mathcal{G}$ is a tree

Under technical conditions, the SDP relaxation has a rank one solution

- QCQP on acyclic graph [BGML12]
- Geometry and Pareto-front [ZT11]

QCQP on acyclic graph

Consider problem (**P**) to find $x \in \mathbb{C}^n$ that

$$\begin{array}{ll} \text{minimize} & x^* C x \\ \text{subject to} & x^* C_k x \leq b_k, k = 1, \ldots, n \end{array}$$

where $C \succeq 0$ and $C_k = C_k^*$ not necessary positive semidefinite.

QCQP on acyclic graph

Define graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ corresponding to the problem **P**:

$$\mathcal{V} = \{1, \ldots, n\}$$

$$\mathcal{E} = \{(i, j) : (i \neq j) \text{ and } ([C_k]_{ij} \neq 0 \text{ for some } k)\}.$$

Theorem: if

(a) $\mathcal{G}$ is connected and acyclic,

(b) for any edge $(i, j) \in \mathcal{E}$, the origin is not in the relative interior of the convex hull of $\{C_{ij}, [C_k]_{ij}, k = 1, \ldots, n\}$,

(c) the set of feasible solution of **P** is bounded and has a strictly feasible point,

then QCQP **P** can be solved in polynomial time.

QCQP on acyclic graph

Applying the result to OPF with the network being a tree,

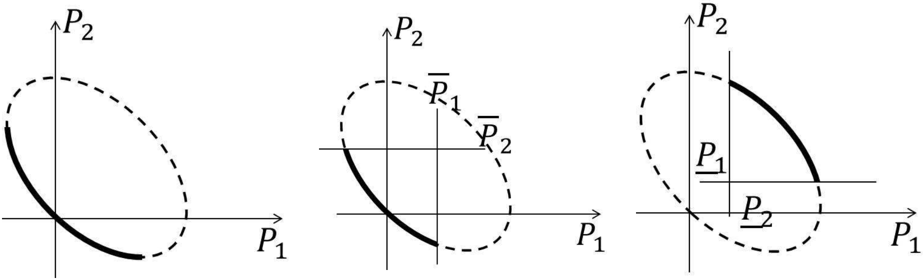
- condition (a) holds
- condition (b) requires removing lower bounds for certain real and reactive power constraints.

Geometry and Pareto-front

- Recall the objective depends only on real power, is separable, convex, quadratic and increasing.
- The set of Pareto-optimal points contains the minimizer.
- Denote the feasible set for real power as $P = \{p \in \mathbb{R}^n : \underline{v}_j \leq |v_j| \leq \overline{v}_j, \underline{p}_j \leq p_j \leq \overline{p}_j, \underline{q}_j \leq q_j \leq \overline{q}_j, p_j + iq_j = v_j(Yv)_j^*, j \in \mathcal{V}\}$
- If $\mathcal{O}(P) = \mathcal{O}(\mathrm{convhull}(P))$, solving the optimization over the convex hull of the (nonconvex) feasible set of OPF problem yields the solution of OPF.

Definition: Let $A \subset \mathbb{R}^n$. A point $x \in A$ is said to be a Pareto-optimal point if there does not exist another point $\tilde{x} \in A$ such that $\tilde{x} < x$. Denote the set of Pareto-optimal points of A as $\mathcal{O}(A)$. The set $\mathcal{O}(A)$ is also called Pareto-front.

Geometry and Pareto-front



Geometry and Pareto-front

Theorem: Consider a tree network with n buses. If

(a) If $i \sim k$, then either $\underline{p}_i = -\infty$ or $\underline{p}_k = -\infty$.

(b) $\underline{q}_i = -\infty$ for all i.

Then $\mathcal{O}(P) = \mathcal{O}(\mathrm{convhull}(P))$.

SDP relaxation has a rank one solution if the conditions are satisfied.

Transmission network: $\mathcal{G}$ has cycles

Theorem [MSL13]: Consider a weakly-cyclic network with cycles of size 3. The following statements hold:

(a) The SDP relaxation has a rank one solution in the lossless case $(\mathrm{Re}(Y) = 0)$, provided $\underline{q}_j = -\infty$ for every $j \in \mathcal{V}$.

(b) The SDP relaxation has a rank one solution in the lossy case, provided $\underline{p}_j = \underline{q}_j = -\infty$ for every $j \in \mathcal{V}$.

???

- Even if the SDP relaxation may have a rank 1 solution, the solver may fail to find it
  - SDP relaxation may have more than one solutions, and some of them have higher rank
  - Algorithms to find low rank solutions among the set of solutions of SDP relaxation?
- What if the SDP does not has a rank 1 solution?
  - What if I get a W of rank 2? Is it totally useless?
  - Heuristic for solving the problem? Heuristic with provable guarantees ($ALG \leq \alpha OPT$ for fixed $\alpha > 1$)?

References

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Thank you!