CME350Q: The ABCs of TQC

Willie Aboumrad

Contents

Part I

Introduction

Chapter I

QM/QC basics

The advent of computers in the second half of the last century brought about a technological revolution that has shaken society to its core. Quantum computing promises to bring about the next era in the revolution, as hypothetical quantum computers have been shown to be able to solve problems far beyond the reach of the classical computers of today.

Computation is a mechanical process. It is the act of transforming information encoded in a physical system by physically manipulating the system. Consider, for instance, how the beads in each column of an abacus represent a digit in the decimal representation of a number, and how we may add a given quantity to some representation by moving the beads in each column.

In order to abstract the mechanical process of computation, we need to quantify information. The basic unit is a *bit*, which represents either a 0 or a 1. The classical computers of today encode bits in current-carrying wires and they process information by directing the current through transistors, which are devices designed to combine the current (or lack thereof) in two input wires into a single output in a predictable fashion. The transistors modify the current value in each wire, thereby implementing logical transformations on the encoded information.

Quantum computation is any computational model based on the theoretical ability to manufacture, manipulate, and measure quantum states. In this course we will explore Topological Quantum Computing (TQC) as a means of constructing quantum computers that are fault-tolerant at the hardware level.

I.1 The postulates of quantum mechanics

We begin with a brief discussion/review of the postulates of quantum mechanics (QM). QM is a mathematical framework for the development of physical theories: on its own, it does not tell you what laws a system must obey, but rather it provides a mathematical and conceptual framework for the development of such laws. The postulates of QM, which were derived after a long process of trial and error, are listed below. The motivation the postulates is not always clear, and the postulates seem surprising even to experts [14].

1. *State space.* There is a complex Hilbert space (complex vector space equipped with an inner product), known as the *state space*, associated to any physical system. Just like in classical mechanics, a quantum system possesses a state at any moment. In particular, the state is a unit vector $|v\rangle$ in the associated Hilbert space, which is the collection of all possible states. Two non-zero vectors represent the same state if and only if $|v_1\rangle = \lambda |v_2\rangle$ for some $\lambda \neq 0$. The linear structure of a Hilbert space implements the *superposition* principle: if $|\psi_1\rangle$ and $|\psi_2\rangle$ denote distinct states, the system may very well be in the state

$$\alpha |\psi_1\rangle + \beta |\psi_2\rangle$$

   at some point. In quantum computing we use the ordinary finite-dimensional Hilbert space $\mathbb{C}^n$.

2. *Evolution.* The evolution of a closed quantum system is described by a unitary transformation. That is, the state $|\psi\rangle$ of the system at a time t_1 is related to the state $|\psi'\rangle$ at a later time t_2 by a *unitary* operator U ($U^\dagger U = (\bar{U}^T)U = I$) depending only on the times t_1 and t_2, so

$$|\psi'\rangle = U |\psi\rangle .$$

   In particular, the system evolves under Schrödinger dynamics:

$$i\hbar \frac{\partial |\psi\rangle}{\partial t} = H |\psi\rangle .$$

   Here, $i = \sqrt{-1}$, $\hbar$ denotes *Planck's constant*, whose value is determined experimentally, and H denotes the system's *Hamiltonian*. The Hamiltonian must be *Hermitian* ($H^\dagger = H$), so

$$|\psi_t\rangle = e^{\frac{-i}{\hbar}tH} |\psi_0\rangle$$

is related to the initial state $|\psi_0\rangle$ by the unitary operator $e^{\frac{-i}{\hbar}tH}$. In quantum computing, we apply unitary transformations to a state $|\psi\rangle$ that physically encodes information.

3. *(Projective) Measurement.* Measurement of a quantum system is given by a Hermitian operator M, called an *observable.* Since M is Hermitian, its eigenvalues are real and they describe the possible measurement outcomes: observables are the quantum analogue of measurable quantities, like position, momentum, and energy, in classical physics. Every observable M has a spectral decomposition
$$M = \sum_j \lambda_j P_j,$$
with P_j denoting the projector onto the eigenspace of M with eigenvalue λ_j. If each eigenspace is 1-dimensional, we say that the measurement is *complete.* Upon measuring a normalized state $|\psi\rangle$ with a complete measurement operator M, the probability of observing the result λ_j is
$$p(\lambda_j) = \langle\psi|\, P_j \,|\psi\rangle\,.$$
Given that the outcome λ_j occurred, the state of the quantum system immediately after the measurement is
$$\frac{P_j\,|\psi\rangle}{\sqrt{p(j)}},$$
the (normalized) eigenvector with eigenvalue λ_j. If we let $|e_j\rangle$ denote an orthonormal basis of eigenvectors of M, we can write $|\psi\rangle = \sum_j a_j\,|e_j\rangle$ for some complex a_j. Then $p(\lambda_j) = |a_j|^2$ and the system will be in state $|e_j\rangle$ if the measurement result is λ_j. Note that measurement interrupts the deterministic unitary evolution and outputs a random variable $X : \{e_j\} \to \{\lambda_j\}$ with distribution $\mathbb{P}(X = \lambda_j) = |a_j|^2$. This is the source of randomness in quantum mechanics. In quantum computing, the possible states are strings of 0's and 1's and measurement produces a *read-out* from the computation.

4. *Composite systems.* If two systems with Hilbert spaces V_1 and V_2 are brought together, then the state of the joint system is $V_1 \otimes V_2$. Composite systems have *entangled* states, which *cannot* be written as the tensor product of single-system states. For instance, consider a quantum system, such as the spin of an electron, described by the Hilbert space $\mathbb{C}^2$. Let $|0\rangle$ and $|1\rangle$ denote distinct states, such as the possible results of measuring the spin in the x-direction. If

there are two interacting electrons, then the state of the composite system is $\mathbb{C}^2 \otimes \mathbb{C}^2$ and the entangled state

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

cannot be expressed as a product $|a\rangle \otimes |b\rangle$ of any single-spin states $|a\rangle$ and $|b\rangle$. In the last equality, $|00\rangle$ denotes $|0\rangle \otimes |0\rangle$ and similarly $|11\rangle = |1\rangle \otimes |1\rangle$.

I.2 Qubits

While classical bits model on-off switches, quantum bits, or *qubits*, model 2-level quantum systems. Mathematically, the states of a bit are $\mathbb{Z}_2 = \{0, 1\}$, while the states of a qubit belong to the state space $\mathbb{C}^2$ equipped with the standard sesquilinear inner product. In particular, we work with an orthonormal basis of $\mathbb{C}^2$ labeled by $|0\rangle$, $|1\rangle$. We call this the *computational basis*.

In this course, we will work with the abstract model of a qubit, as described by the postulates in Section I.1, rather than consider any one physical system that encodes it in the real world. In this case, the Hilbert space is $\mathbb{C}^2$, the evolution operators belong to the group $U(2)$ of 2×2 unitary matrices, and measurements are given by 2×2 Hermitian operators.

A qubit state is a non-zero vector $|\psi\rangle$ in $\mathbb{C}^2$. Since any $|\psi'\rangle = \lambda |\psi\rangle$ with $\lambda \neq 0$ represents the same physical state, a qubit state is really an equivalence class of vectors, i.e. a point on the Riemann sphere or complex projective line $P^1(\mathbb{C})$. In quantum mechanics, $P^1(\mathbb{C})$ is also known as the Bloch sphere, which provides a neat visualization for quantum states. We will often use the term *qubit* to mean qubit state.

The reader should question whether a qubit state $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$ carries the same information as a classical bit. Each complex coefficient is described by two real numbers, and each real number is in turn described by an infinite binary string. Thus, it would seem that a single quantum bit in fact encodes infinitely many classical bits. However, this is misleading, because if we *measure* the qubit, the state may take exactly one of two possible values: $|0\rangle$ or $|1\rangle$. This uniquely quantum phenomenon is known as *wave-function collapse* and it explains that we may only ever extract *one* bit of classical information from a qubit.

It follows from the Postulate 4 in Section I.1 that a composite system of n interactive qubits has state space $(\mathbb{C}^2)^{\otimes n}$. In this case, there is a preferred orthonormal basis, or computational basis, labeled by the n-bit strings $|0 \cdots 00\rangle, |0 \cdots 01\rangle, \ldots, |1 \cdots 1\rangle$.

Part II

Mathematical interlude

Chapter II

Braids, links, and tangles

II.1 Braids

Braids will play a critical role in our study of anyon theory. We begin with a rigorous definition.

Let I denote closed interval $[0,1]$ in $\mathbb{R}$. We say that any topological space homeomorphic to $I=[0,1]$ is a *topological space*.

Definition II.1.1. [10, Definition 1.4] A (geometric) braid on $n \geq 1$ strands is a set $b \subset \mathbb{R}^2 \times I$ formed by disjoint topological intervals called strands such that the projection $\mathbb{R}^2 \times I \to I$ maps each string homeomorphically onto I and

$$\begin{aligned} b \cap (\mathbb{R}^2 \times \{0\}) &= \{(1,0,0),(2,0,0),\ldots,(n,0,0)\} \\ b \cap (\mathbb{R}^2 \times \{1\}) &= \{(1,0,1),(2,0,1),\ldots,(n,0,1)\}. \end{aligned}$$

Remark II.1.2. Braids can also be defined using piece-wise linear strands with no local extrema with respect to the "height" projection $\mathbb{R}^2 \times I \to I$. For more details, see [8, Definition X.6.1].

Two geometric braids b and b' on n strands are *isotopic* if b can be continuously deformed into b' in the class of braids. More formally, b and b' are isotopic if there is a continuous map $F : b \times I \to \mathbb{R}^2 \times I$ such that for each $s \in I$, the map $F_s : b \to \mathbb{R}^2 \times I$ sending $x \in b$ to $F(x,s)$ is an embedding whose image is a geometric braid on n strands, $F_0 = \mathrm{id}_b : b \to b$, and $F_1(b) = b'$ [10]. Note that each F_s automatically maps every endpoint of b to itself. The map F and the family of braids $\{F_s(b)\}_{s\in I}$ are called an *isotopy* of $b = F_0(b)$ into $b' = F_1(b)$.

Clearly, the notion of isotopy is an equivalence relation on the class of geometric braids on n strands.

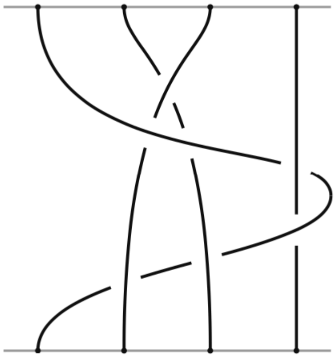


Figure II.1: A braid on 4 strands.

Much like a pair of loops, two braids can be multiplied by concatenation. In particular, if $b_1, b_2 \subset \mathbb{R}^2 \times I$ are two geometric braids on n strands, then the product $b_1 b_2$ is the set of points $(x, y, t) \in \mathbb{R}^2 \times I$ such that $(x, y, 2t) \in b_1$ if $0 \leq t \leq 1/2$ and $(x, y, 2t-1) \in b_2$ if $1/2 \leq t \leq 1$ [10].

We will usually deal with *braid diagrams*, rather than with geometric braids directly. In order to specify a geometric braid, we can draw its projection to $\mathbb{R} \times \{0\} \times I$ along the second coordinate and specify which strand goes "under" the other one at each crossing point, as in Figure II.1. We restrict consideration only to those braids whose projection to $\mathbb{R} \times \{0\} \times I$ includes only double transversal crossings. For more details, see [10, Section 1.2.2].

Conventions We parse each diagram from bottom to top, and by convention the $(i+1)$st strand in σ_i crosses over the ith. Figure II.4 shows the braid relation in B_3. Each text uses its own convention. For instance, [10] reads diagrams from top to bottom. Later in this course we will interpret the vertical direction as the time axis, so we prefer thinking that time increases like the z-axis, as we move upwards.

Definition II.1.3. The *braid group* B_n on n strands is the free group generated by $\sigma_1, \ldots, \sigma_{n-1}$ subject to the *braid relations*,

$$\begin{aligned} \sigma_i \sigma_j &= \sigma_j \sigma_i, \quad |i-j| > 1 \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}. \end{aligned} \tag{1.1}$$

The braid group B_n models isotopy classes of braids on n strands, so it has a nice diagrammatic interpretation. For instance, the braid generator σ_2 is drawn in Figure II.2, for $n = 4$.

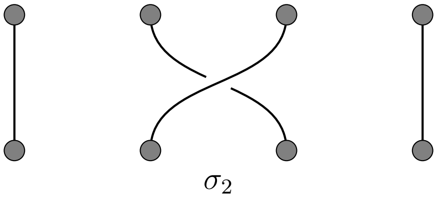


Figure II.2: Braid generator $\sigma_2 \in B_4$.

Multiplication in the group is achieved by vertical concatenation of the corresponding diagrams. For instance, Figure II.3 illustrates the product $\sigma_2\sigma_1$. The identity braid has diagram given by n parallel strands.

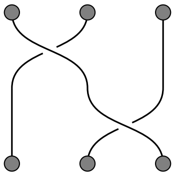


Figure II.3: Multiplying two braid diagrams. Here is the product $\sigma_2\sigma_1$ in B_3.

Notice that each braid on n strands defines a permutation in the symmetric group S_n. In particular, an n strand braid defines a map $\{1, 2, \ldots, n\} \times \{0\} \times \{0\} \to \{s(1), s(2), \ldots, s(n)\} \times \{0\} \times \{1\}$, so we have a homomorphism $B_n \to S_n$. For example, the braid in Figure II.1 defines the permutation $(2, 3)$, given in cycle notation. The kernel of this map is known as the group of *pure braids*. For more details, see [10, Section 1.3].

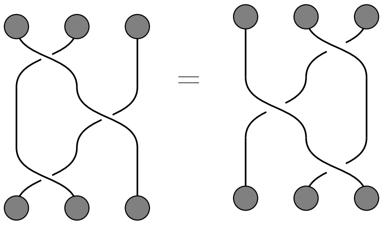


Figure II.4: The braid relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ in B_3.

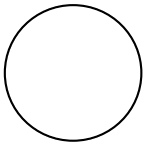
II.2 Knots and links

A *link* in $\mathbb{R}^3$ is a one-dimensional smooth compact submanifold of $\mathbb{R}^3$, in other words, a disjoint union of embedded circles. A *knot* is a connected link. A link is *oriented* if each of its connected components is oriented. Two oriented links L and L' are *equivalent* or *ambient isotopic* if there exists an orientation-preserving diffeomorphism f of $\mathbb{R}^3$ such that $f(L) = L'$.

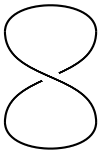
We will also need *framed links*, which come equipped with a non-zero section of the normal bundle, also up to positive diffeomorphism. We may suggestively think of a framed link as a link made of ribbons or embedded bands. The framing keeps track of "twists" in the band: in contrast to a string, we can give each "side" of the ribbon a different color. (We disallow "half-twists" to avoid the case of forming non-orientable Möbius bands.) By convention, framed links will be drawn with the *blackboard framing*, so that the framing lies in the plane of the page. Two oriented framed links are equivalent if they are equivalent as oriented links and the isotopy preserves the framing.

Every (framed) link can be pictured using a diagram in two dimensions by projecting perpendicularly onto a plane, chosen so that the projection has only transversal crossings, none of which have more than two branches [8, Proposition X.3.4]. If we think of the direction perpendicular to the chosen plane as "vertical," the branch of the (framed) link which lies "below" the branch it crosses is shown broken in the diagram. Thus we say every (framed) link has a *regular projection*. Arrows in the diagram indicate orientation.

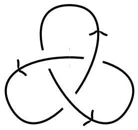
A. Unknot



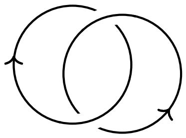
B. Framed unknot



C. Or. trefoil



D. Or. Hopf Link



The equivalence class of the link L can be recovered from a regular link projection $\bar{L}$. A great theorem of Reidemeister states that link projections define equivalent links if and only if they can be obtained from each other by applying an isotopy of the plane together with a finite sequence of local *Reidemeister moves*, depicted in Figure II.5 [8, Theorem X.3.7]. (See [16] for the original 1927 statement.) Thus equivalence under plane isotopy and Reidemeister moves I, II, and III coincides with ambient isotopy.

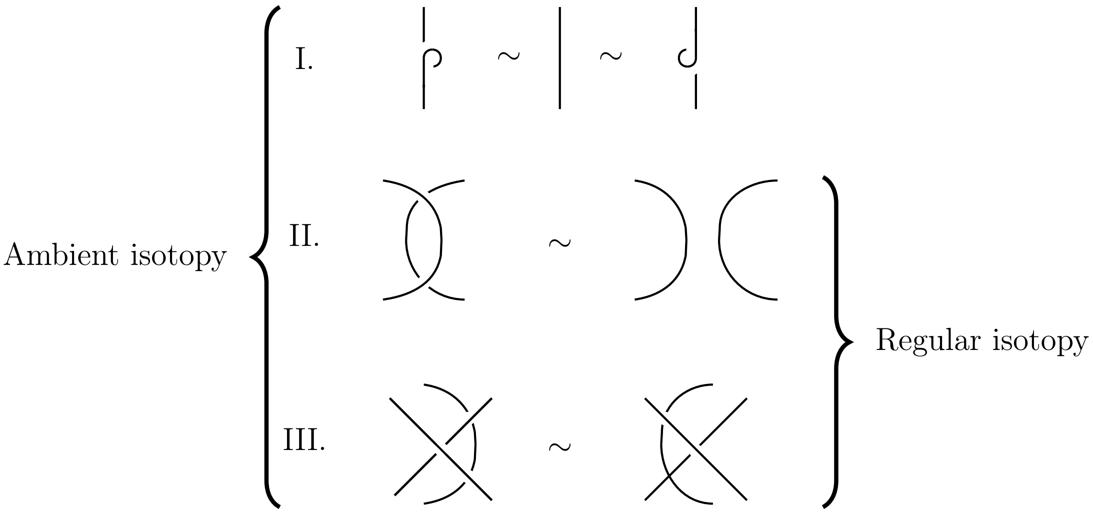


Figure II.5: Reidemeister moves.

There is also a framed version of Reidemeister's theorem. Regular projections of *framed* links define equivalent framed links if and only if they can be obtained from each other by applying an isotopy of the plane and a finite sequence of Reidemeister moves II and III, and the modified Reidemeister move I' pictured in Figure II.8.

Figure II.6: Reidemeister move I on the core of an embedded band twists the ribbon.

It is easy to see that Reidemeister moves II and III on the cores of embedded bands extend to the bands bands themselves, while move I does not extend [11]. Rather, move I corresponds to a full twist of the band, as depicted in Figure II.6: the normal vector field turns by an angle of 2π or -2π. Each diagram T_+ and T_-, depicted in Figure II.7, is obtained by changing the framing of the diagram $\downarrow$. The modified Reidemeister move I' suggests that twisting the band by 2π and then by -2π, or vice versa, leaves the band invariant.

Reidemeister's theorem is not particularly useful at distinguishing links in practice. For this we turn to a *link invariant*, defined on (oriented) link projections, as a function which remains unchanged by Reidemeister moves I, II, and III.

Figure II.7: On the left is the diagram T_+. On the right, T_-.

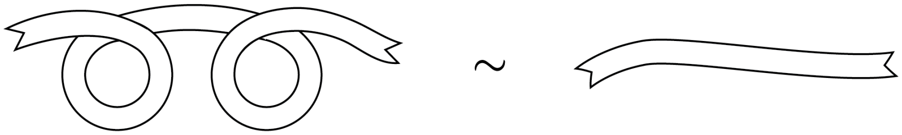


Figure II.8: Modified Reidemeister move I'.

There are also invariants of *framed* (oriented) links. Since ambient isotopy of framed links preserves the framing, equivalence of framed (oriented) links corresponds to the notion of *regular isotopy*, which is the equivalence class of links obtained by an isotopy of the plane combined with a finite sequence of Reidemeister moves II and III [11]. Thus an invariant of framed (oriented) links is an invariant of regular isotopy.

Remark II.2.1. Kauffman defines the term *regular isotopy* in [11] motivated by the fact that Reidemeister moves II and III can be seen as regular homotopies of immersed circles, but move I has no such interpretation, since contraction of the loop would violate differentiability.

The following lemma shows that certain invariants of framed links become invariants of (oriented) links, independent of framing, when suitably normalized.

We will use X_+ and X_- to denote positive and negative crossing diagrams, as shown in Figure II.9. The *writhe* $w(\bar{L})$ of the oriented link projection $\bar{L}$ is the sum of the signs of all the crossings, where the crossing signs are $+1$ for X_+ and -1 for X_-.

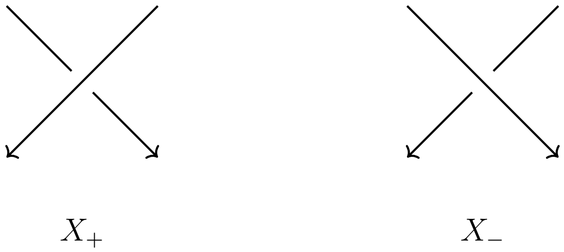


Figure II.9: Positive and negative crossing diagrams.

The *self-writhe* $\psi(\bar{L})$ of $\bar{L}$ is the sum of the writhes of each component of L. Equivalently, $\psi(\bar{L})$ is defined as the sum of the crossing signs of self-crossings of components of L [11]. Note that $\psi(\bar{L})$ is independent of orientation, since the sign of a crossing does not change if we change the orientation of both branches at the crossing.

Lemma II.2.2. *[11, Lemma 2.1] Let A be a ring and let a be an invertible element of A. Suppose $R(\bar{L})$ is a regular isotopy invariant of oriented link diagrams $\bar{L}$ satisfying*

$$\begin{aligned} R(T_+) &= aR(\downarrow), \text{ and} \\ R(T_-) &= a^{-1}R(\downarrow), \end{aligned}$$

with T_+ and T_- as in Figure II.7. Then $S(\bar{L}) = a^{-w(\bar{L})}R(\bar{L})$ is an invariant of ambient isotopy for oriented links.

Proof. By inspecting the diagrams, we see that $w(\bar{L})$ is an invariant of regular isotopy. Therefore S is an invariant of regular isotopy, as the product of two such invariants.

Hence we need only show $S(\bar{L})$ is invariant under move I. This follows immediately from the definition of $S(\bar{L})$, since

$$w(T_+) = 1 + w(\downarrow) \quad \text{and} \quad w(T_-) = -1 + w(\downarrow).$$

□

Remark II.2.3. If we normalize using the *self-writhe* instead, we obtain an invariant for *unoriented* links [11]. In particular, if V is an invariant of regular isotopy for oriented links which is affected in the same way as R by a Reidemeister move I, then $U(\bar{L}) = a^{-\psi(\bar{L})}V(\bar{L})$ is an invariant of ambient isotopy for unoriented links. The claim follows by noticing that the self-writhe is affected by the Reidemeister moves in exactly the same way as the writhe $w(\bar{L})$.

There is a special class of link invariants which can be calculated using an inductive procedure. Three oriented link projections $\bar{L}_+$, $\bar{L}_-$, $\bar{L}_0$ are said to be *skein related* if they coincide outside a disk in $\mathbb{R}^2$ and are respectively isotopic inside the disk to X_+, X_- and $\downarrow\downarrow$, with X_+ and X_- as in Figure II.9. Given a link projection $\bar{L}$, "resolve" $\bar{L}$ by first "changing" and "smoothing" some crossing in $\bar{L}$, and then again changing and smoothing a crossing in each of the two projections that result from $\bar{L}$, and so on until only unlinks remain [4]. "Changing" a crossing amounts to replacing X_+ by X_- and vice versa, and "smoothing" smoothing a crossing replaces it with $\downarrow\downarrow$. The invariant is then computed by fixing its value on the unknot and using a skein relation.

One such invariant is the *HOMFLY polynomial* P.

Theorem II.2.4. *[4] There is a unique function P the from the set of oriented links to the set of Laurent polynomials in two variables a, z such that $P = 1$ for the unknot and*

$$aP_{\bar{L}_+} - a^{-1}P_{\bar{L}_-} = zP_{\bar{L}_0}, \tag{2.1}$$

whenever $\bar{L}_+, \bar{L}_-$, and $\bar{L}_0$ are skein related.

Remark II.2.5. Here we use the skein relation proposed by Jones, since it more conveniently fits our setting [6, Proposition 6.2]. It is also preferred in [11] and Theorem 15.1.1 of [3].

The HOMFLY polynomial is named after the four groups of authors, Freyd and Yetter, Hoste, Lickorish and Millet, and Ocneanu, who discovered it independently and submitted to the AMS within a few days of each other [4]. Przytycki and Traczyk also discovered the invariant independently a few months later, but the invariant had already been named [15].

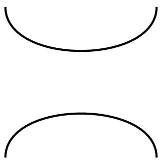
There is also a framed version of the HOMFLY polynomial, which is defined by the skein relation

$$\tilde{P}_{\bar{L}_+} - \tilde{P}_{\bar{L}_-} = z\tilde{P}_{\bar{L}_0}, \tag{2.2}$$

along with $\tilde{P}(T_\pm) = a^{\pm 1}\tilde{P}(\downarrow)$ and a chosen "initial value" for the unknot [11].

The *Kauffman polynomial* K can also be evaluated using the inductive "splicing" procedure facilitated by a skein relation. In what follows we denote by $\bar{L}_\infty$ a regular link projection of L which is homeomorphic inside a disk to the diagram in Figure II.10.

Theorem II.2.6. *[11, Theorem 2.3] There exists a unique function $\tilde{K}$ from the set of framed unoriented links to the set of Laurent polynomials in two variables a, z*



$\bar{L}_\infty$

Figure II.10: A projection of a link L homoemorphic inside a disk to the above diagram is denoted $\bar{L}_\infty$.

such that $\tilde{K} = 1$ *for the unknot,* $K(T_+) = aK$ *and* $\tilde{K}(T_-) = a^{-1}\tilde{K}$*, and the following identity holds whenever* $\bar{L}_+, \bar{L}_-, \bar{L}_0$*, and* $\bar{L}_\infty$ *are skein related,*

$$\tilde{K}_{\bar{L}_+} - \tilde{K}_{\bar{L}_-} = z(\tilde{K}_{\bar{L}_0} - \tilde{K}_{\bar{L}_\infty}).$$

We define $\tilde{K}$ *on oriented links by forgetting the framing.*

While K is only an invariant of regular isotopy and is not invariant under Reidemeister move I, we use Lemma II.2.2 and Remark II.2.3 to renormalize $\tilde{K}$ to obtain an invariant K of (un)oriented links, independent of framing. Unfortunately, K does not satisfy a convenient skein relation.

Remark II.2.7. Kauffman calls $\tilde{K}$ the *Dubrovnik variant* of his eponymous polynomial, after the city in which he discovered it [11]. Lickorish noted that the Dubrovnik variant can be obtained from Kauffman's original polynomial by scaling the parameters and normalizing the result by an appropriate root of unity [11, Section 7].

Knots are closely related to braids. The connection is facilitated by *Alexander's theorem*. If β is a braid, we let $\hat{\beta}$ denote its *closure*, which is obtained by joining the corresponding top and bottom endpoints in the braid diagram of β with simple non-intersecting arcs, as represented in the middle diagram of Figure II.12. Note that a braid has a canonical orientation, where each strand is pointing downward, so that its closure is also oriented.

Theorem II.2.8 (Alexander)**.** *[1] Up to equivalence, every oriented link diagram is the closure of some braid.*

Although it is clear that equivalent braids have equivalent closures, the converse is false. *Markov's theorem* states the precise conditions under which closed braids define equivalent oriented links. To state it, we must first observe that there are obvious embeddings $B_n \hookrightarrow B_{n+1}$ of the braid groups defined in II.1.3, given by adding an $(n+1)$th strand to a braid $\beta \in B_n$ sufficiently far away from the strands of β so it is not linked to any of them, as depicted in Figure II.11. We say that two braids β, β' in $B_\infty = \coprod_n B_n$ are *Markov equivalent* if they can be obtained from each other by applications of the braid relations (1.1) and a finite sequence of *Markov moves*:

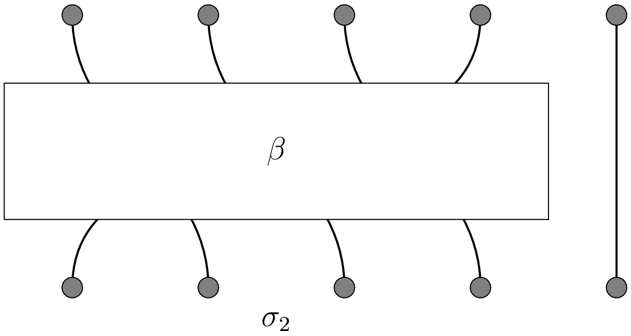


Figure II.11: Embedding $\beta \in B_n$ into B_{n+1}.

$$\begin{aligned} \beta' &\sim \beta\beta'\beta^{-1} \quad (\beta, \beta' \in B_n) \\ \beta &\sim \beta\sigma_n^{\pm 1} \qquad (\beta \in B_n). \end{aligned}$$

Here σ_n denotes the nth braid generator in B_{n+1}, as in Definition II.1.3.

Closing the braid β and a conjugate of β by a single braid generator results in two links which differ by Reidemeister moves II and III, which take place in the closure of β, as in the left diagram of Figure II.12 [12]. Similarly, adding a new strand which crosses once over or under the last strand of β corresponds in the closure to a Reidemeister move I. This *stabilization move* is depicted in the right diagram of Figure II.12.

Theorem II.2.9 (Markov). *[2, Theorem 2.3] Two braids have equivalent closures if and only if they are Markov equivalent.*

Remark II.2.10. Though first stated in 1935, this theorem was first rigorously proved in 1974, as Theorem 2.3 in [2].

Since the stabilization move in Figure II.12 corresponds to a Reidemeister move I on the closure, any two links which are regularly isotopic must be the closure of

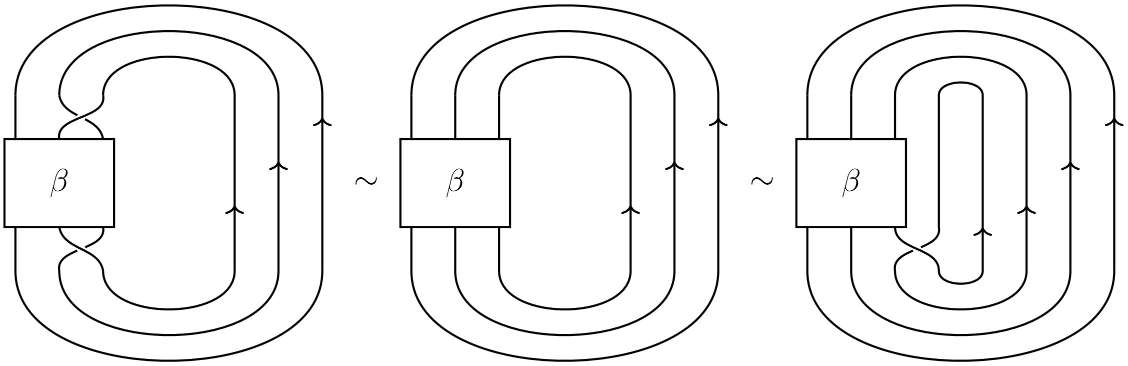


Figure II.12: From left to right, the closure of a conjugate of $\beta \in B_n$ by a single braid generator, the closure $\hat{\beta}$ of β, and the closure of $\beta\sigma_n^{-1} \in B_{n+1}$.

braids which can be obtained from each other by applications of the braid relations (1.1) and a sequence of conjugation moves only. Regular isotopy corresponds to ambient isotopy of framed links, so two braids $\beta, \beta' \in B_n$ close to the same *framed* link if and only if they can be obtained from each other by the braid relations and conjugation by braid generators $\sigma_i \in B_n$.

Using Markov's theorem, we construct oriented link invariants via functionals on Markov equivalence classes in B_∞. Vastly generalizing the pioneering work of Jones in [7], we associate to every finite-dimensional module over any ribbon Hopf algebra H an invariant of oriented framed links by considering the categorical trace. When V is simple, we obtain an invariant independent of framing by renormalizing as in Lemma II.2.2.

II.3 Tangles

The skein relation defining the Kauffman invariant includes the diagram $\bar{L}_\infty$, depicted in Figure II.10, which does not quite fit into our picture of links as closed braids. Recall that the "time slice" for any $0 \leq t \leq 1$ intersects a braid diagram on n strands at exactly n points. So with its local extrema, $\bar{L}_\infty$ does not correspond to any braid but it ought to come from a braid-like object [11]. We must consider more general objects in order to bring $\bar{L}_\infty$ into our algebraic framework. Enter *tangle diagrams.*

Definition II.3.1. Let k and ℓ be nonnegative integers. A *tangle* T of type (k, ℓ) is the union of a finite number of pairwise disjoint simple oriented curves in $\mathbb{R}^2 \times [0,1]$ such that the boundary ∂T of T satisfies

$$\partial T = T \cap (\mathbb{R}^2 \times [0,1]) = ([k] \times \{0\} \times \{0\}) \cup ([\ell] \times \{0\} \times \{1\}),$$

with $[n] = \{1, 2, \dots, n\}$ for $n > 0$ and $[0] = \emptyset$. *Framed tangles* or *ribbons* come equipped with a non-zero section of the normal bundle, much like framed links.

A *tangle diagram* is a projection of a tangle T onto $\mathbb{R} \times [0,1]$ which preserves the boundary of T. We let $\mathcal{FT}_n$ denote the $\mathbb{C}$-algebra of type (n,n) framed tangle diagrams generated by formal linear combinations of diagrams, with multiplication defined by vertical concatenation as in the case of braids.

Note that every link is a type $(0,0)$ tangle and that every type $(0,0)$ tangle defines a link up to isotopy.

We take all tangle diagrams up to isotopy of the plane. Two tangles are *equivalent* if they have the same boundary and a diffeomorphism of $\mathbb{R}^2 \times [0,1]$ maps one to the other. Note that two tangle diagrams represent equivalent tangles in $\mathbb{R}^2 \times [0,1]$ if and only if the diagrams can be obtained from each other by a finite sequence of Reidemeister moves I, II, and III [8, Theorem X.5.9]. Tangle diagrams define equivalent framed tangles if they are *regularly isotopic*, which means they are related by plane isotopy and a sequence of Reidemeister moves II and III.

We single out an important subalgebra of tangle diagrams. These tangles bring the $\bar{L}_\infty$ diagrams of the Kauffman skein relation into our algebraic framework. For each $i = 1, \dots, n-1$ let h_i denote the *hook diagram* in $\mathcal{FT}_n$ defined by Figure II.13. Together with the single loop diagram δ, the hook diagrams satisfy the Temperley-Lieb relations (3.1), as shown in Figure II.14. Therefore, the hook diagrams generate the *Temperley-Lieb algebra*.

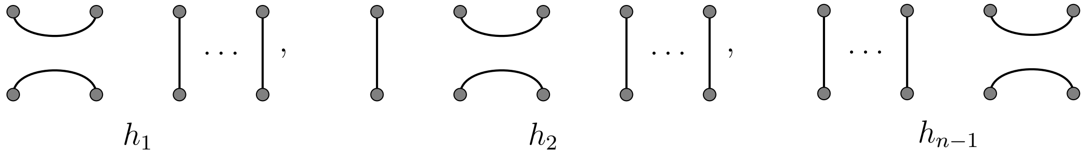


Figure II.13: Hook generators h_i on n strands.

Definition II.3.2. The *Temperley-Lieb algebra* is a $\mathbb{C}[\delta^{\pm 1}]$-algebra generated by $h_1, \dots, h_{n-1}$ subject to the *Temperley-Lieb relations*

$$h_i^2 = \delta h_i, \quad h_i h_{i\pm 1} h_i = h_i, \quad \text{and} \quad h_i h_j = h_j h_i \text{ when } |i-j| > 1. \tag{3.1}$$

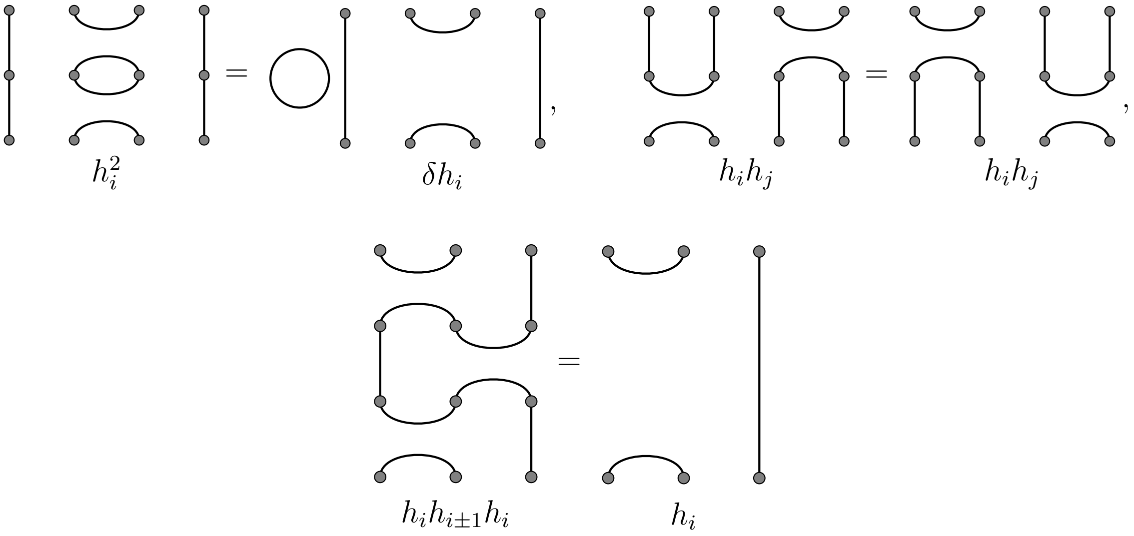


Figure II.14: Hook generators statisfy the Temperley-Lieb relations (3.1). The indices i and j are such that $|i - j| > 1$.

The Kauffman polynomial $\tilde{K}$ is defined on individual tangle elements by evaluating $\tilde{K}$ on the closure of the tangle, and then extended additively to the entire algebra $\mathcal{T}_n$. [11]. Now every link diagram appearing in a calculation of the Kauffman invariant belongs to the subalgebra D_n of $\mathcal{FT}_n$ generated by the braid generators $\sigma_1, \ldots, \sigma_n$ of Definition II.1.3 and the hook generators $h_1, \ldots, h_n$ of Figure II.13.

Chapter III

The ribbon category

So far we have defined a number of topological objects that play prominent roles in the story of anyons. In this chapter we will define an algebraic object capturing the structure of tangles. In particular, we will collect tangles into a diagrammatical category expressing the topological structure. This will be our first example of a *ribbon category*. Later we will model anyon systems using certain ribbon categories.

III.1 $\mathcal{RIBBON}$ and $\mathcal{TANGLE}$

We use categories as an organizing principle. Roughly, a *category* is a collection of objects that are linked by arrows. A category abstracts a notion of entities and the relationships between them. In our context, objects can be thought of as elements of the category and arrows represent morphisms between them. In any category, we can compose arrows *associatively* and every object has an identity arrow pointing to itself. In particular, we have the following definition.

Definition III.1.1. [8, Definition XI.1.1] A *category* $\mathcal{C}$ consists

(1) of a class $\mathrm{Ob}(\mathcal{C})$ whose elements are called the *objects* of the category,

(2) of a class $\mathrm{Hom}(\mathcal{C})$ whose elements are called *morphisms* of the category, and

(3) of maps

$$\begin{aligned} \textit{identity} \quad & \mathrm{id}\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Hom}(\mathcal{C}) \\ \textit{source} \quad & s\colon \mathrm{Hom}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C}) \\ \textit{target} \quad & b\colon \mathrm{Hom}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C}) \end{aligned}$$

$$\textit{composition} \quad \circ\colon \operatorname{Hom}(\mathcal{C}) \times \operatorname{Hom}(\mathcal{C}) \to \operatorname{Hom}(\mathcal{C})$$

such that

(a) for any object $V \in \operatorname{Ob}(\mathcal{C})$, we have

$$s(\mathrm{id}_V) = b(\mathrm{id}_V) = V,$$

(b) for any morphism $f \in \operatorname{Hom}(\mathcal{C})$,

$$\mathrm{id}_{b(f)} \circ f = f \circ \mathrm{id}_{s(f)} = f,$$

(c) for any morphisms f, g, h satisfying $b(f) = s(g)$ and $b(g) = s(h)$, we have

$$(h \circ g) \circ f = h \circ (g \circ f).$$

We first construct the *category of oriented tangle diagrams* $\mathcal{T}$ as follows [8, XIV.5.1]. The objects of $\mathcal{T}$ are finite sequences of ± signs, including the empty sequence, and the morphisms are isotopy classes of oriented tangle diagrams, as in Definition II.3.1. Given an oriented tangle T of type (k, ℓ), we define the source and target by

$$\operatorname{dom}(T) = (\epsilon_1, \ldots, \epsilon_k) \quad \text{and} \quad \operatorname{codom}(T) = (\eta_1, \ldots, \eta_\ell),$$

where $\epsilon_i = +$ (resp. $\eta_i = +$) if the point $(i, 0, 0)$ (resp. the point $(i, 0, 1)$) is an endpoint (resp. an origin) of T. We have $\epsilon_i = -$ and $\eta_i = -$ in the remaining cases. If T is of type $(0, \ell)$ or $(k, 0)$ then $\operatorname{dom}(T)$ or $\operatorname{codom}(T)$ is the empty sequence. For each sequence ϵ of length n, the identity id_ϵ is the isotopy class of the tangle formed by the union of invervals $[n] \times 0 \times [0, 1]$, with orientation defined by $\operatorname{dom}(\mathrm{id}_\epsilon) = \operatorname{codom}(\mathrm{id}_\epsilon) = \epsilon$, and we take $\mathrm{id}_\emptyset$ to be the empty tangle. Composition is of course composition of tangles, and $T_1 \circ T_2$ is defined only when $\operatorname{codom}(T_2) = \operatorname{dom}(T_1)$.

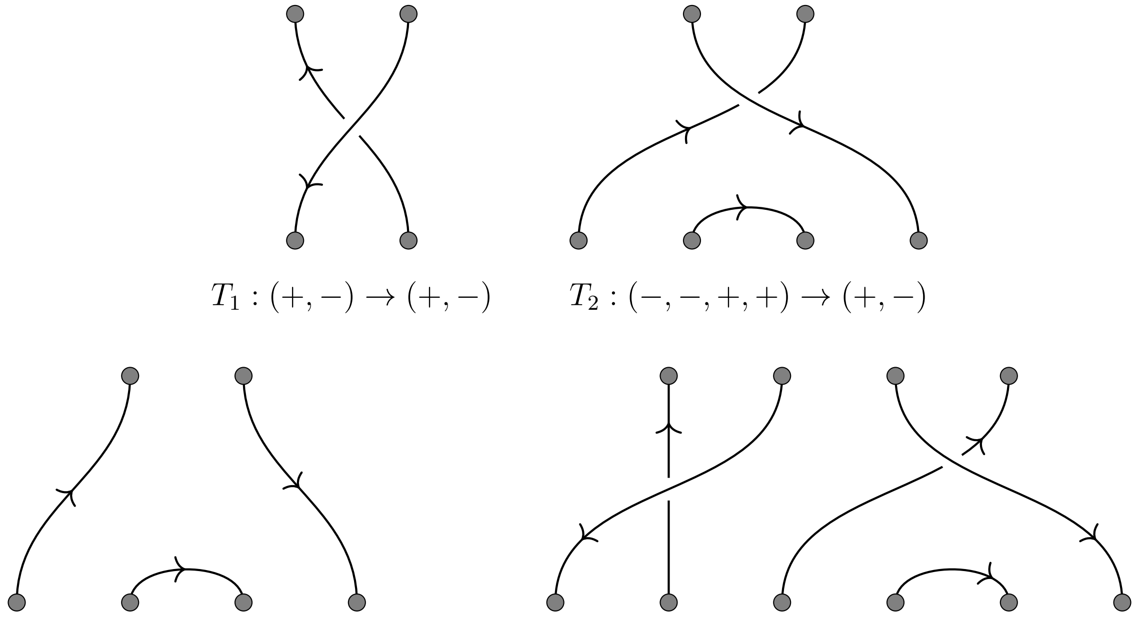
The category $\mathcal{RIBBON}$ of framed tangles or ribbons is obtained in exactly the same way, using bands instead of strands, and taking the morphisms only up to regular isotopy.

We may equip $\mathcal{T}$ with a tensor product, defined on objects by concatenation of sequences and on morphisms by horizontal concatenation of diagrams. In particular, if $\epsilon = (\epsilon_1, \ldots, \epsilon_k)$ and $\epsilon' = (\epsilon_{k+1}, \ldots, \epsilon_{k+\ell})$ are objects then

$$\epsilon \otimes \epsilon' = (\epsilon_1, \ldots, \epsilon_k, \epsilon_{k+1}, \ldots, \epsilon_{k+\ell})$$

and the tensor product $T_1 \otimes T_2$ of tangle diagrams is the diagram obtained by placing T_2 to the right of T_1, as in Figure III.1.

The tensor product $\otimes$ gives $\mathcal{T}$ the structure of a *tensor* or *monoidal* category.



$T_1 : (+,-) \to (+,-)$

$T_2 : (-,-,+,+) \to (+,-)$

$T_1 \circ T_2$

$T_1 \otimes T_2$

Figure III.1: Composition and tensor product of tangles.

Definition III.1.2. [13, Section VII.1] A *monoidal category* $(\mathcal{C}, \otimes, I, a, l, r)$ is a category $\mathcal{C}$ with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a unit object I, and three natural isomorphisms a, l, r.

This means that

(a) we have an object $V \otimes W$ associated to any pair (V, W) of objects in the category,

(b) we have a morphism $f \otimes g$ associated to any pair (f, g) of morphisms of $\mathcal{C}$ such that

$$s(f \otimes g) = s(f) \otimes s(g) \quad \text{and} \quad b(f \otimes g) = b(f) \otimes b(g),$$

(c) if f' and g' are morphisms such that $s(f') = b(f)$ and $s(g') = b(g)$, then

$$(f' \otimes g') \circ (f \otimes g) = (f' \circ f) \otimes (g' \circ g),$$

(d) and

$$\mathrm{id}_{V \otimes W} = \mathrm{id}_V \otimes \mathrm{id}_W.$$

The *associativity constraint* a for $\otimes$ is a *natural isomorphism*

$$a\colon \otimes(\otimes \times \mathrm{id}) \to \otimes(\mathrm{id} \times \otimes).$$

This means that for any triple (U, V, W) in $\mathcal{C}$, there exists an isomorphism

$$a_{U,V,W}\colon (U \otimes V) \otimes W \to U \otimes (V \otimes W)$$

such that the square

$$\begin{tikzcd}
(U \otimes V) \otimes W \arrow[r, "a_{U,V,W}"] \arrow[d, "(f\otimes g)\otimes h"] & U \otimes (V \otimes W) \arrow[d, "f\otimes(g\otimes h)"] \\
(U' \otimes V') \otimes W' \arrow[r, "a_{U',V',W'}"] & U' \otimes (V' \otimes W')
\end{tikzcd} \tag{1.1}$$

commutes whenever f, g, h are morphisms in the category. The associativity constraint a must satisfy the Pentagon Axiom, encoded in the commutativity of the diagram (P), for all triples (U, V, W) of objects in $\mathcal{C}$.

$$\begin{tikzcd}
\big(U \otimes (V \otimes W)\big) \otimes X \arrow[dd, "a_{U,V\otimes W,X}"'] & \big((U \otimes V) \otimes W\big) \otimes X \arrow[l, "a_{U,V,W}\otimes \mathrm{id}_X"'] \arrow[d, "a_{U\otimes V,W,X}"] \\
 & (U \otimes V) \otimes (W \otimes X) \arrow[d, "a_{U,V,W\otimes X}"] \\
U \otimes \big((V \otimes W) \otimes X\big) \arrow[r, "\mathrm{id}_U \otimes a_{V,W,X}"'] & U \otimes \big(V \otimes (W \otimes X)\big)
\end{tikzcd} \tag{P}$$

The *left unit constraint* [resp. *right unit constraint*] is a natural isomorphism

$$l\colon \otimes(I \otimes \mathrm{id}) \to \mathrm{id} \qquad [\text{resp. } r\colon \otimes(\mathrm{id} \otimes I) \to \mathrm{id}].$$

This means that for any object V of $\mathcal{C}$ there exists an isomorphism

$$l_V\colon I \otimes V \to V \qquad [\text{resp. } r\colon V \otimes I \to V]$$

such that

$$\begin{tikzcd}
I \otimes V \arrow[r, "l_V"] \arrow[d, "\mathrm{id}_V\otimes f"] & V \arrow[d, "f"] \\
I \otimes V' \arrow[r, "l_{V'}"] & V'
\end{tikzcd} \qquad \left[\text{resp.} \quad \begin{tikzcd}
V \otimes I \arrow[r, "r_V"] \arrow[d, "f\otimes \mathrm{id}_V"] & V \arrow[d, "f"] \\
V' \otimes I \arrow[r, "r_{V'}"] & V'
\end{tikzcd}\right]$$

commutes for any morphism $f\colon V \to V'$.

In addition, the associativity constraint must be compatible with the unit constraints l, r, so we require the Triangle Axiom, encoded in the commutativity of the diagram (T) for all pairs (V, W) of objects in $\mathcal{C}$.

$$\begin{tikzcd}
(V \otimes I) \otimes W \arrow[rr, "a_{V,I,W}"] \arrow[dr, "r_V \otimes \mathrm{id}_W"'] & & V \otimes (I \otimes W) \arrow[dl, "\mathrm{id}_V \otimes l_W"] \\
& V \otimes W &
\end{tikzcd} \tag{T}$$

The tensor category is said to be *strict* if the associativity and unit constraints a, l, r are identities of the category.

The (strict) monoidal category of (framed) tangle diagrams has a convenient presentation in terms of generators and relations.

Theorem III.1.3. *[18, Lemmas 5.2,5.3] Every oriented framed tangle diagram is the composition of tensor products of the eight tangles X_+, X_-, $\overset{\rightarrow}{\cap}$, $\overset{\leftarrow}{\cap}$, $\underset{\rightarrow}{\cup}$, $\underset{\leftarrow}{\cup}$, $\downarrow$, and $\uparrow$, and the composition is completely determined by the monoidal structure and the relations in Figure III.2. The same is true for unoriented framed tangles if we forget the orientation in the generators and the relations. The same is also true for (oriented) tangles, if we also impose equality after Reidemeister move I, as depicted in Figure II.5.*

Proof (sketch). That the set G of given morphisms generates the set of diagrammatic tangles can be observed by dividing an arbitrary tangle diagram into horizontal strips, such that each strip involves only a crossing or a local maximum or minimum. Thus to every tangle we may associate a word in the alphabet G.

For details on the sufficiency of the relations see Theorem 3.5 in [5] or XII.3 in [8]. □

Theorem III.1.3 provides a purely algebraic presentation of $\mathcal{RIBBON}$. It will allow us to show that $\mathcal{RIBBON}$ is universal in the sense that for any *ribbon category* $\mathcal{C}$ we can construct a functor $\mathcal{F}\colon \mathcal{RIBBON} \to \mathcal{C}$ simply by specifying the values of $\mathcal{F}$ on each $\mathcal{RIBBON}$ generator and verifying that the relations in Figure III.2 hold in the image.

We now turn to exploring the additional structure present in $\mathcal{RIBBON}$ that leads to the definition of a *ribbon category*. First, both $\mathcal{T}$ and $\mathcal{RIBBON}$ are equipped with a *braiding*, making them *braided categories*.

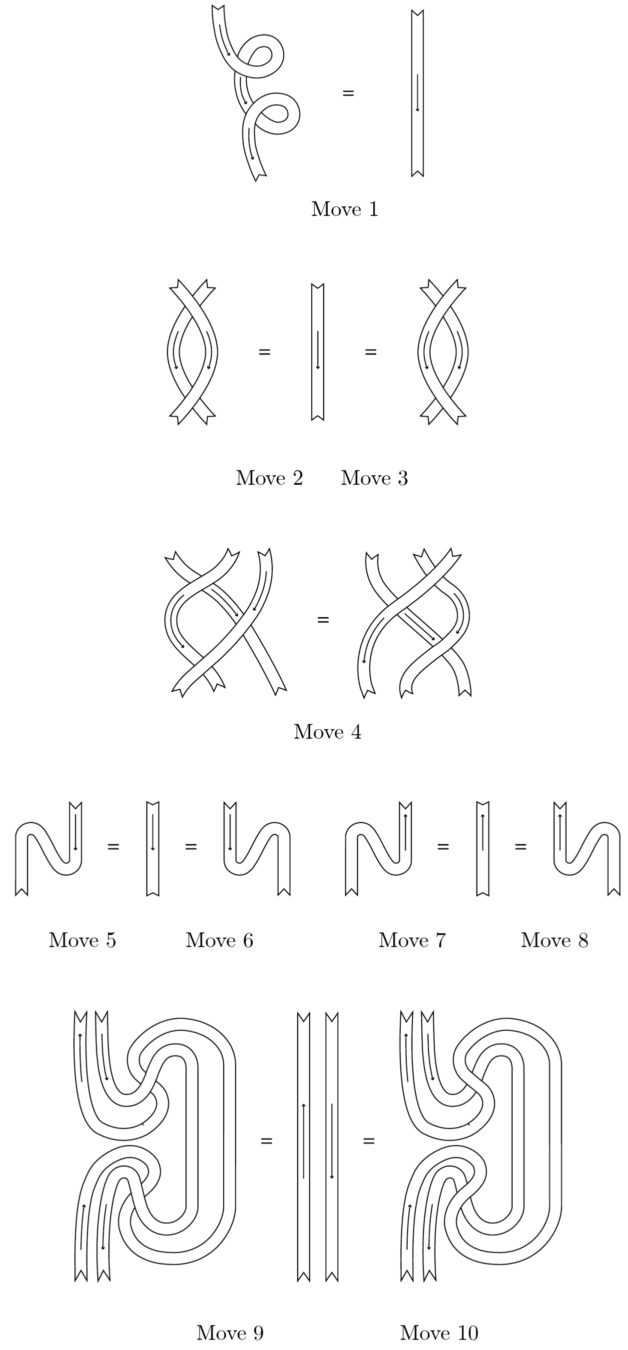


Figure III.2: Relations between generators in RIBBON.

Definition III.1.4. Let $(\mathcal{C}, \otimes, I, a, l, r)$ be a monoidal category with tensor product $\otimes$, unit object I, associativity constraint a, left unit constraint l, and right unit constraint r. Then $\mathcal{C}$ is a *braided tensor category* if there exists a commutativity constraint c such that for any pair of objects V, W the map

$$c_{V,W} : V \otimes W \to W \otimes V$$

is a natural isomorphism. In addition, we require that c satisfies the Hexagon Identities, which are encoded in the commutativity of the diagrams (H1) and (H2) below.

$$\begin{tikzcd}
& U \otimes (V \otimes W) \arrow[r, "c_{U,V\otimes W}"] & (V \otimes W) \otimes U \arrow[dr, "a_{V,W,U}"] & \\
(U \otimes V) \otimes W \arrow[ur, "a_{U,V,W}"] \arrow[dr, "c_{U,V}\otimes \mathrm{id}_W"'] & & & V \otimes (W \otimes U) \\
& (V \otimes U) \otimes W \arrow[r, "a_{V,U,W}"'] & V \otimes (U \otimes W) \arrow[ur, "\mathrm{id}_V \otimes c_{U,W}"'] &
\end{tikzcd} \tag{H1}$$

$$\begin{tikzcd}
& (U \otimes V) \otimes W \arrow[r, "c_{U\otimes V,W}"] & W \otimes (U \otimes V) \arrow[dr, "a^{-1}_{W,U,V}"] & \\
U \otimes (V \otimes W) \arrow[ur, "a^{-1}_{U,V,W}"] \arrow[dr, "\mathrm{id}_U \otimes c_{V,W}"'] & & & (W \otimes U) \otimes V \\
& U \otimes (W \otimes V) \arrow[r, "a^{-1}_{U,W,V}"'] & (U \otimes W) \otimes V \arrow[ur, "c_{U,W}\otimes \mathrm{id}_V"'] &
\end{tikzcd} \tag{H2}$$

We call c a *braiding* in $\mathcal{C}$ and we say $\mathcal{C}$ is *braided*. If the braiding satisfies

$$c_{W,V} \circ c_{V,W} = \mathrm{id}_{V \otimes W} \tag{1.2}$$

for all objects V, W in $\mathcal{C}$, then c is called a *symmetric braiding* and $\mathcal{C}$ is a *symmetric tensor category*.

For any pair of objects ϵ, ϵ' the braiding operator $c_{\epsilon,\epsilon'} : \epsilon \otimes \epsilon' \to \epsilon' \otimes \epsilon$ of $\mathcal{T}$ is depicted in Figure III.3.

The braiding operators of any braided category satisfy the *Yang-Baxter equation.* holds in $\mathrm{End}(V \otimes V \otimes V)$.

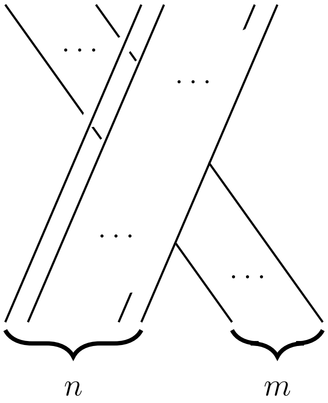


Figure III.3: Braiding structure in $\mathcal{T}$.

Lemma III.1.5 (Yang-Baxter equation)**.** *For any objects U, V, W in a strict braided category $\mathcal{C}$, the following equation holds in* $\mathrm{End}(U \otimes V \otimes W)$*:*

$$(c_{V,W} \otimes \mathrm{id}_U)(\mathrm{id}_V \otimes c_{U,W})(c_{U,V} \otimes \mathrm{id}_W) = (\mathrm{id}_W \otimes c_{U,V})(c_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes c_{V,W}).$$

Proof. Using (H2), we have that

$$(c_{V,W} \otimes \mathrm{id}_U)(\mathrm{id}_V \otimes c_{U,W}) = c_{V \otimes U,W}$$

and similarly

$$(c_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes c_{V,W}) = c_{U \otimes V,W}.$$

Combining with the naturality of the braiding operators yields the desired result:

$$\begin{aligned}(c_{V,W} \otimes \mathrm{id}_U)(\mathrm{id}_V \otimes c_{U,W})(c_{U,V} \otimes \mathrm{id}_W) &= c_{V \otimes U,W}(c_{U,V} \otimes \mathrm{id}_W)\\ &= (\mathrm{id}_W \otimes c_{U,V})c_{U \otimes V,W}\\ &= (\mathrm{id}_W \otimes c_{U,V})(c_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes c_{V,W}).\end{aligned}$$

□

The tangle diagram category also has *left duality.*

Definition III.1.6. [8, Definition XIV.2.1] Let $(\mathcal{C}, \otimes, I)$ be a strict tensor category with unit object I. It is a tensor category with *left duality* if for each object V of $\mathcal{C}$ there exists and object V^* and morphisms

$$b_V : I \to V \otimes V^* \quad \text{and} \quad d_V : V^* \otimes V \to I$$

subject to the *rigidity axioms*

$$(\mathrm{id}_V \otimes d_V)(b_V \otimes \mathrm{id}_V) = \mathrm{id}_V, \quad \text{and} \quad (d_V \otimes \mathrm{id}_{V^*})(\mathrm{id}_{V^*} \otimes b_V) = \mathrm{id}_{V^*}. \tag{1.3}$$

For any morphism $f : V \to W$, we define the *transpose* $f^* : W^* \to V^*$ of f by the formula

$$f^* = (d_W \otimes \mathrm{id}_{V^*})(\mathrm{id}_{W^*} \otimes f \otimes \mathrm{id}_{V^*})(\mathrm{id}_{W^*} \otimes b_V). \tag{1.4}$$

There is a similar notion of a tensor category with *right duality*.

Definition III.1.7. A strict tensor category $(\mathcal{C}, \otimes, I)$ has *right duality* if for each object V of $\mathcal{C}$ there exists an object ${}^*V$ and morphisms

$$b'_V \colon I \to {}^*V \otimes V \quad \text{and} \quad d'_V \colon V \otimes {}^*V \to I$$

satisfying

$$(d'_V \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes b'_V) = \mathrm{id}_V \quad \text{and} \quad (\mathrm{id}_{{}^*V} \otimes d'_V)(b'_V \otimes \mathrm{id}_{{}^*V}) = \mathrm{id}_{{}^*V} \tag{1.5}$$

Again we may extend the map $V \to {}^*V$ to a functor by defining the morphism ${}^*f\colon {}^*V \to {}^*W$ for each $f\colon V \to W$ by

$${}^*f = (\mathrm{id}_{{}^*V} \otimes d'_W)(\mathrm{id}_{{}^*V} \otimes f \otimes \mathrm{id}_{{}^*W})(b'_V \otimes \mathrm{id}_{{}^*W}.$$

Right duality has properties analogous to those enjoyed by left duality. In general, however, right duality is different from left duality unless we add additional hypotheses on $\mathcal{C}$. For example, if $\mathcal{C}$ is a *ribbon category* (see below), then the right and left duals of an object V coincide. In general, it may happen that $\mathcal{C}$ is *autonomous*, which means that it has both right and left duality. In this case, there are always automorphisms

$${}^*(V^*) \cong V \cong ({}^*V)^*$$

for every object V [8, Section XIV.2].

Given a sequence $\epsilon = (\epsilon_1, \dots, \epsilon_k)$ define the dual object ϵ^* by the sequence $(-\epsilon_1, \dots, -\epsilon_k)$ and the *birth* and *death* maps $b_\epsilon : \emptyset \to \epsilon \otimes \epsilon^*$ and $d_\epsilon : \epsilon^* \otimes \epsilon \to \emptyset$ by the tangles represented in Figure III.4. Observe that the transpose T^* of a tangle T, defined by (1.4), is isotopic to the tangle obtained by reflecting the drawing through its horizontal midsection [19]. This (correctly) suggests that if $f\colon V \to W$ and $g\colon U \to V$ are morphisms in $\mathcal{T}$, then $(f \circ g)^* = g^* \circ f^*$.

Theorem III.1.8. *The categories $\mathcal{T}$ and $\mathcal{RIBBON}$ are strict braided tensor categories with left duality, in the sense of Definition III.1.4 and Definition III.1.6.*

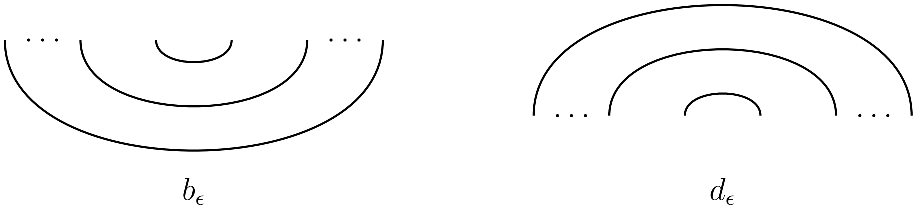


Figure III.4: Birth and death maps equipping $\mathcal{T}$ with a left duality.

Proof. The proof is by diagrams. It is clear that the birth and death maps b_ϵ and d_ϵ satisfy the rigidity axioms 1.3. To show c is a braiding we must verify it satisfies the Hexagon Identities (H1) and (H2). The former is shown in Figure III.5 and the latter is proven similarly. ☐

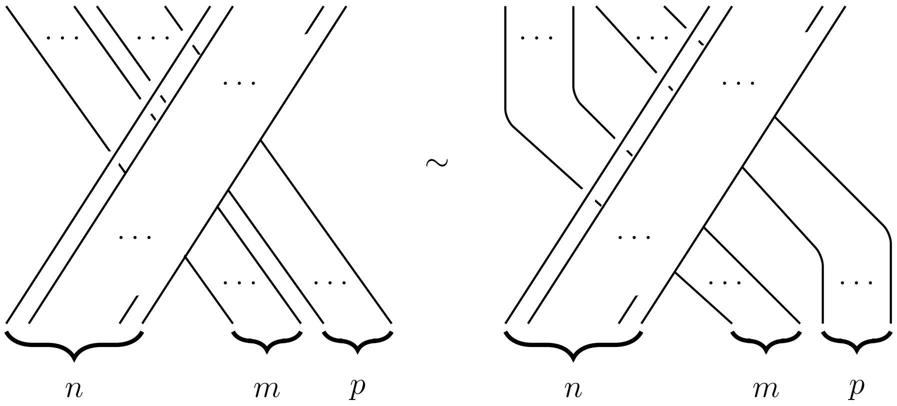


Figure III.5: Braiding in $\mathcal{T}$ satisfies the Hexagon Identity (H1).

Suggestively named, $\mathcal{RIBBON}$ is also equipped with a family of *twists* θ.

Definition III.1.9. A strict braided tensor category $\mathcal{C}$ with left duality in the sense of Definition III.1.4 and III.1.6 is a *ribbon category* if there exists a family $\theta_V : V \to V$ of natural isomorphisms, called *twists*, such that

$$\theta_{V\otimes W} = (\theta_V \otimes \theta_W)c_{V,W}c_{W,V} \quad \text{and} \quad \theta_{V^*} = (\theta_V)^*. \tag{1.6}$$

The transpose f^* of a morphism f is defined by Relation (1.4).

We define $\theta_{(+)}$ as the left ribbon T_+ in Figure II.7, oriented downwards. Its inverse is the right ribbon T_- in Figure II.7, also oriented downwards. The twist is determined for arbitrary objects by the defining relations 1.6. Thus the twist θ_ϵ corresponds to the tangle obtained by rotating the identity tangle id_ϵ by an angle of 2π about the vertical axis bisecting the diagram. We have the following theorem.

Theorem III.1.10. *[18, Theorem 4.6] The category of ribbons $\mathcal{RIBBON}$ is a ribbon category in the sense of Definition III.1.9.*

Note that we can use the braiding and the twist in a ribbon category $\mathcal{C}$ to equip $\mathcal{C}$ with right duality. Set ${}^*V = V^*$ and define the morphisms

$$b'_V = (id_{V^*} \otimes \theta_V) c_{V,V^*} b_V \quad \text{and} \quad d'_V = d_V c_{V,V^*} (\theta_V \otimes \mathrm{id}_{V^*}).$$

In sharp contrast to an arbitrary monoidal category, a ribbon category allows for a consistent assignment of dimensions for objects and traces of morphisms [9].

Definition III.1.11. [8, Defintion XIV.4.1] Consider any ribbon category with unit object I. For any object V and any endomorphism f of V, the *categorical trace* $\mathrm{tr}_q(f)$ of f is the element of $\mathrm{End}(I)$ given by

$$\mathrm{tr}_q(f) = d'_V (f \otimes \mathrm{id}_{V^*}) b_V = d_V c_{V,V^*} (\theta_V f \otimes \mathrm{id}_V{}^*) b_V.$$

The *categorical dimension* $\dim_q(V)$ of V is defined as the element

$$\dim_q(V) = \mathrm{tr}_q(\mathrm{id}_V) = d'_V b_V$$

of $\mathrm{End}(I)$.

The naturality of the braiding gives the alternative expression

$$\mathrm{tr}_q(f) = d_V c_{V,V^*} (\theta_V f \otimes \mathrm{id}_{V^*}) b_V = d_V (\mathrm{id}_{V^*} \otimes \theta_V f) c_{V,V^*} b_V = d_V (\mathrm{id}_{V^*} \otimes f) b'_V.$$

The categorical trace shares many properties with the usual matrix trace. In particular, the categorical trace in the category $\mathrm{Vec}_f(k)$ of finite-dimensional vector spaces over the field k coincides with the usual matrix trace. Writing $v = \sum_i (v^i, v) v_i$ for dual bases $\{v_i\}$ and $\{v^i\}$ of k-vector spaces V and V^*, respectively, we see that

$$\mathrm{tr}_q(f) = \sum_i d_V \tau \circ (f \otimes \mathrm{id}_{V^*})(v_i \otimes v^i) = \sum_i (v^i, f(v_i)) = \mathrm{tr}\{f\}.$$

Finally, we arrive at the universality result for the ribbon category $\mathcal{RIBBON}$.

Theorem III.1.12. *[17, Theorem 2.5] Let $\mathcal{C}$ be a ribbon category and let V be an object of $\mathcal{C}$. Then there exists a unique covariant tensor functor $\mathcal{F}_V : \mathcal{RIBBON} \to \mathcal{C}$ preserving the braiding, the left duality, and the twist, such that $\mathcal{F}_V(+) = V$ and $\mathcal{F}_V(-) = V^*$, and*

$$\mathcal{F}_V(\downarrow) = \mathrm{id}_V, \quad \mathcal{F}_V(\uparrow) = \mathrm{id}_{V^*}, \quad \mathcal{F}_V(X_+) = c_{V,V}, \quad \mathcal{F}_V(X_-) = c_{V,V}^{-1}$$
$$\mathcal{F}_V(\overrightarrow{\cap}) = d_V, \quad \mathcal{F}_V(\overleftarrow{\cap}) = d'_V, \quad \mathcal{F}_V(\underset{\rightarrow}{\cup}) = b_V, \quad \mathcal{F}_V(\underset{\leftarrow}{\cup}) = b'_V.$$

Proof. It suffices to check that the maps $\mathrm{id}_V, \mathrm{id}_{V^*}, c_{V,V}, c_{V,V}^{-1}, d_V, d'_V, b_V, b'_V$ satisfy the relations in Theorem III.1.3. Moves 1–4 are simply restatements of the Reidemeister moves I', II, and III, which correspond to the invertibility of the twist, invertibility of the braiding, and the Yang-Baxter equation **??**, respectively. Moves 5–8 represent the rigidity axioms 1.3 and their counterparts 1.5. Moves 9–10 encode the defining axioms (1.6) of the twist. Verifying them requires additional relations like

$$c_{V^*,W} = (d_V \otimes \mathrm{id}_{W\otimes V^*})(\mathrm{id}_{V^*} \otimes c_{V,W}^{-1} \otimes \mathrm{id}_{V^*})(\mathrm{id}_{V^*\otimes W} \otimes b_V),$$

which is depicted by Figure III.6. The additional relations required to show that

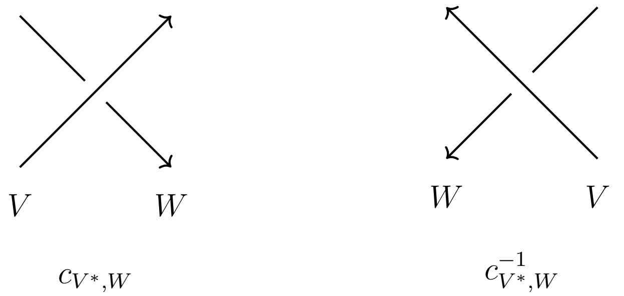


Figure III.6: $c_{V^*,W}$ and its inverse.

Moves 9 and 10 hold in the image of $\mathcal{F}$ are consequences of Proposition XIV.3.1 in [8]. ☐

The crowning achievement of Theorem III.1.12 in our context is that it provides a means of constructing link invariants. As a type $(0,0)$ tangle, the link L is mapped by $\mathcal{F}_V$ to an object in $\mathrm{End}(I)$ via any of its projections $\bar{L}$, where I denotes the unit object in $\mathcal{C}$. Theorem III.1.12 proves that any other link regularly isotopic to L is mapped to the same object in $\mathrm{End}(I)$. For example, when $\mathcal{C}(\mathfrak{g})$ is the category of finite-dimensional modules over a quasi-triangular Hopf algebra like $U_q(\mathfrak{g})$, $\mathrm{End}(I) \cong \mathbb{C}(q)$, and we obtain numerical invariants of framed links.

Proposition III.1.13. *[17, Lemma 2.6] Let T be a type (k,k) framed oriented tangle which is an endomorphism of an object ϵ in $\mathcal{RIBBON}$. Let $\bar{L}$ be the framed link diagram obtained by closing T. For any object V of the ribbon category $\mathcal{C}$ we have*

$$\mathcal{F}_V(\bar{L}) = \mathrm{tr}_q(\mathcal{F}_V(T)).$$

Proof. The proof is straightforward, obtained by combining definitions into a single diagram. Set $V^\epsilon = V^{\epsilon_1} \otimes \cdots \otimes V^{\epsilon_k}$ with $V^+ = V$ and $V^- = V^*$. By Definition III.1.11, we have

$$\mathrm{tr}_q(\mathcal{F}_V(T)) = d'_{V^\epsilon}(\mathcal{F}_V(T) \otimes \mathrm{id}_{(V^\epsilon)^*}) b_{V^\epsilon}.$$

Using the definitions in Figure III.4 and the uniqueness of $\mathcal{F}_V$, it follows that $\mathrm{tr}_q(\mathcal{F}_V(T))$ is exactly the image of the closure of T, as in Figure III.7. ☐

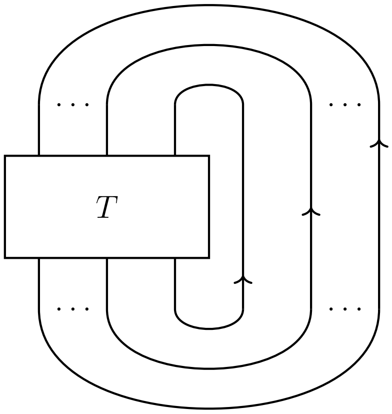


Figure III.7: Categorical trace as the closure of the tangle $T : (+, \ldots, +) \to (+, \ldots, +)$.

Bibliography

[1] J. W. Alexander. A lemma on systems of knotted curves. *Proceedings of the National Academy of Sciences*, 9(3):93–95, Jan 1923.

[2] Joan S. Birman. Braids, links, and mapping class groups. (am-82). 1975.

[3] Vyjayanthi Chari and Andrew Pressley. *A guide to quantum groups.* Cambridge University Press, 1994.

[4] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millett, and A. Ocneanu. A new polynomial invariant of knots and links. *Bulletin of the American Mathematical Society*, 12(2):239–247, Jan 1985.

[5] Peter J Freyd and David N Yetter. Braided compact closed categories with applications to low dimensional topology. *Advances in Mathematics*, 77(2):156–182, 1989.

[6] V. F. R. Jones. Hecke algebra representations of braid groups and link polynomials. *The Annals of Mathematics*, 126(2):335, 1987.

[7] Vaughan F. R. Jones. A polynomial invariant for knots via von Neumann algebras. *Bulletin of the American Mathematical Society*, 12(1):103–111, Jan 1985.

[8] Christian Kassel. *Quantum Groups.* Springer-Science, first edition, 1995.

[9] Christian Kassel, Marc Rosso, and Vladimir Georgievi˘ c Turaev. *Quantum groups and knot invariants.* Société mathématique de France, 1997.

[10] Christian Kassel and Vladimir G. Turaev. *Braid groups.* Springer, 2010.

[11] Louis H. Kauffman. An invariant of regular isotopy. *Transactions of the American Mathematical Society*, 318(2):417–471, Jan 1990.

[12] Sofia Lambropoulou. Diagrammatic representations of knots and links as closed braids, 2018, arXiv:1811.11701.

[13] Saunders Mac Lane. *Categories for the working mathematician.* Springer, 1988.

[14] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information.* Cambridge University Press, 2021.

[15] Jozef H. Przytycki and Pawel Traczyk. Invariants of links of conway type. 2016, arXiv:1610.06679.

[16] Kurt Reidemeister. Elementare begründung der knotentheorie. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 5(1):24–32, 1927.

[17] N. Reshetikhin and V. G. Turaev. Invariants of 3-manifolds via link polynomials and quantum groups. *Inventiones Mathematicae*, 103(1):547–597, 1991.

[18] N. Y. Reshetikhin and V. G. Turaev. Ribbon graphs and their invaraints derived from quantum groups. *Communications in Mathematical Physics*, 127(1):1–26, 1990.

[19] David N. Yetter. Markov algebras. *Braids Contemporary Mathematics*, pages 705–730, 1988.