Chapter 2

Mixed and saddle point problems

2.1 Abstract mixed problems

The main results presented in this Section date back to the seminal works by I. Babuška [1] and F. Brezzi [3]. Many textbooks offer a comprehensive presentation of the underlying theory. In particular, the reader is encouraged to consult [7] and [6] which have inspired the authors of these lecture notes. Here, the main objective is to provide a brief and self-contained set of notes that provide the reader with the basic tools needed for understanding the difficulties encountered in the finite element solution of problems such as those based on the Stokes or Darcy (see Sections 1.3.1 and 1.3.2) equations, or elasticity problems with nearly incompressible materials (Section 1.3.3).

Let $\Big(X, \|\cdot\|_X\Big)$ and $\Big(M, \|\cdot\|_M\Big)$ be two real Hilbert spaces equipped with the scalar products $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_M$. Their dual spaces are denoted as usual by X' and M'. Two continuous bilinear forms are introduced, $a : X \times X \to \mathbb{R}$ and $b : X \times M \to \mathbb{R}$ associated with the operators $A : X \to X'$ and $B : X \to M'$ defined by

$$\langle Au, v\rangle = a(u, v), \quad \forall\,(u, v) \in X \times X, \tag{2.1}$$

$$\langle Bv, q\rangle = b(v, q), \quad \forall\,(v, q) \in X \times M. \tag{2.2}$$

The dual operator of B is denoted by B^T and is defined as $B^T : M \to X'$, $\langle B^T q, v\rangle = b(v, q) = \langle Bv, q\rangle$, for all $(v, q) \in X \times M$.

For $f \in X'$ and $g \in M'$, the following problem is considered.

Find $(u,p) \in X \times M$ *such that, for all* $(v,q) \in X \times M$

$$\left\{ \begin{array}{rcl} a(u,v) + b(v,p) & = & \langle f, v\rangle, \\ b(u,q) & = & \langle g, q\rangle. \end{array} \right. \tag{2.3}$$

The above problem can also be written as

Find $(u,p) \in X \times M$ *such that*

$$\left\{ \begin{array}{rcl} Au + B^T p & = & f, \\ Bu & = & g. \end{array} \right. \tag{2.4}$$

Next, necessary and sufficient conditions for this problem to have a unique solution are sought after. For this purpose, the kernel of B,

$$V = \mathrm{Ker}\, B = \{u \in X, Bu = 0\} = \{u \in X, b(u,q) = 0, \forall\, q \in M\}, \tag{2.5}$$

is introduced along with its polar set,

$$V^\circ = (\mathrm{Ker}\, B)^\circ = \{h \in X', \langle h, w\rangle = 0, \forall\, v \in V\}.$$

Note that the polar set is a kind of "orthogonal" space for the duality pairing $\langle \cdot, \cdot \rangle$.

Let Π denote the canonical injection of X' into V'. For $h \in X'$, define Πh in V' by

$$\langle \Pi h, v\rangle = \langle h, v\rangle, \text{ for all } v \in V.$$

In other words, if h is a continuous linear form defined on X then Πh is its restriction to V. Note that $\|\Pi h\|_{V'} \le \|h\|_{X'}$ and that $V^\circ = \mathrm{Ker}\, \Pi$.

Theorem 2.1
Problem (2.4) has a unique solution if and only if

- *(i)* $\Pi \circ A$ *is an isomorphism from* $V = \mathrm{Ker}\, B$ *onto* $V' = (\mathrm{Ker}\, B)'$.
- *(ii)* $B : X \to M'$ *is surjective.*

Proof.

$\boxed{\Rightarrow}$ First, suppose that (2.4) has a unique solution. In this case, the surjectivity of B can be proven as follows. Let $h \in M'$. If problem (2.4) has a unique solution corresponding to $f = 0$ and $g = h$, then there exists $u \in X$ such that $Bu = h$. Next, the surjectivity of $\Pi \circ A$ from V onto V' is proven. For this purpose, let $f \in V'$. From the Hahn-Banach theorem, it follows that the linear continuous form on $V \subset X$, f, can be extended on X. Let $\tilde{f}$ denote this extension (by construction $\Pi \tilde{f} = f$). There exists a unique $(u,p) \in X \times M$ such that

$$\left\{ \begin{array}{rcl} Au + B^T p & = & \tilde{f}, \\ Bu & = & 0. \end{array} \right.$$

Thus, for $v \in V$,

$$\langle Au, v\rangle + \langle B^T p, v\rangle = \langle \tilde{f}, v\rangle = \langle f, v\rangle.$$

Since $\langle B^T p, v\rangle = \langle p, Bv\rangle = 0$, it follows that $\Pi Au = f$.

It remains to show that $\Pi \circ A$ is injective. For this purpose, let $u \in V$ be such that $\Pi Au = 0$. For all $v \in V$,

$$\langle Au, v\rangle = 0, \tag{2.6}$$

and therefore $Au \in V^\circ = (\operatorname{Ker} B)^\circ$.

From the surjectivity of B, it follows that $\operatorname{Im} B = M'$ which is obviously closed in M'. From (2.6) and the closed range theorem (see Remark 2.1), it also follows that $Au \in \operatorname{Im} B^T$. This means that there exists $p \in M$ such that $B^T p = -Au$. Therefore, the pair (u, p) satisfies

$$\left\{ \begin{array}{rcl} Au + B^T p & = & 0, \\ Bu & = & 0. \end{array} \right.$$

Since the above problem is assumed to have a unique solution, this solution is $(0, 0)$. Thus $u = 0$, which proves that $\Pi \circ A$ is injective.

$\boxed{\Leftarrow}$ Next, assume that the statements (i) and (ii) of Theorem 2.1 are true. To prove the existence of u, assumption (ii) is first exploited to define $u_g \in X$ such that $Bu_g = g$. Since $\Pi f - \Pi Au_g$ is an element of V', it follows from assumption (i) that there exists $u_0 \in V$ such that

$$\Pi Au_0 = \Pi f - \Pi Au_g.$$

Hence, if $u = u_0 + u_g$, $\Pi Au = \Pi f$,

$$\langle f - Au, v\rangle = 0, \quad \forall\, v \in V,$$

and therefore $f - Au \in (\operatorname{Ker} B)^\circ$. Since B is surjective, it follows from the closed range theorem that $(\operatorname{Ker} B)^\circ = \operatorname{Im} B^T$. Thus, there exists $p \in M$ such that $f - Au = B^T p$. In conclusion,

$$\begin{array}{rcl} Au + B^T p & = & f, \\ Bu & = & g, \end{array}$$

which proves the existence of a solution. To prove the uniqueness of this solution, consider (u, p) such that

$$\begin{array}{rcl} Au + B^T p & = & 0, \\ Bu & = & 0, \end{array}$$

which implies that $\Pi Au + \Pi B^T p = 0$. Since the restriction of $B^T p$ to V vanishes (see above), then $\Pi Au = 0$. From assumption (i), it follows that $u = 0$. Finally, since B is surjective and B^T is injective, it follows that $B^T p = 0$ implies that $p = 0$, which proves that (2.4) has a unique solution. $\square$

Remark 2.1 (Closed range theorem) *In infinite dimensional spaces, $h \in (Ker\, B)^\circ$ does not in general imply $h \in Im\, B^T$. As a matter of fact, one only has $(Ker\, B)^\circ \supset \overline{Im\, B^T}$. Nevertheles, $(Ker\, B)^\circ = Im\, B^T$ if $Im\, B$ is closed. More precisely, the* closed range theorem *(see* e.g. *H. Brezis [2]) states that the four following propositions are equivalent*

(i) $Im\, B$ *is closed.*

(ii) $Im\, B^T$ *is closed.*

(iii) $(Ker\, B)^\circ = Im\, B^T$.

(iv) $(Ker\, B^T)^\circ = Im\, B$.

Remark 2.2 *Point (ii) of Theorem 2.1 could be replaced by B^T is injective* and *$Im\, B^T$ is closed.*

2.2 Application to the Stokes problem

The above result can be applied to the Stokes problem in Ω with $X = H^1_0(\Omega)^d$, $M = L^2_0(\Omega)$, $A = -\Delta$, $B = -\operatorname{div}$, $B^T = \boldsymbol{\nabla}$, $V = \{\boldsymbol{v} \in X, \operatorname{div} \boldsymbol{v} = 0\}$, and $a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{\nabla u} : \boldsymbol{\nabla v}$, $b(\boldsymbol{v}, q) = -\int_\Omega q \operatorname{div} \boldsymbol{v}$. Assumption (i) of Theorem 2.1 is a direct consequence of the coercivity of $(\boldsymbol{\nabla}\cdot, \boldsymbol{\nabla}\cdot)_0$ on $X \times X$ (see Remark 1.1).

The proof of assumption (ii) is quite complex. The key result is the following one.

Theorem 2.2
Let Ω be an open bounded set with a Lipschitz boundary. Then $\boldsymbol{\nabla} : L^2(\Omega) \to H^{-1}(\Omega)^d$ has a closed range.

The proof of this theorem is very technical. The interested reader can consult [7] (p. 20).

Corollary 2.1
Let Ω be an open bounded and connected set with a Lipschitz boundary. Then the operator div *is surjective from $H^1_0(\Omega)^d$ onto $L^2_0(\Omega)$.*

Proof. The operator $B^T = \boldsymbol{\nabla}$ defined on M has a closed range and is injective (if $\boldsymbol{\nabla} p = 0$ on a connected domain then p is constant, and this constant is zero since it has a zero mean value). Thus the closed range theorem implies that $B = -\operatorname{div}$ is surjective.$\square$

As a consequence, the following existence and uniqueness result can be stated for the Stokes problem.

Corollary 2.2
Let Ω be an open bounded and connected set with a Lipschitz boundary. If $\boldsymbol{f} \in H^{-1}(\Omega)^d$, there exists a unique solution $(\boldsymbol{u}, p) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$ of problem (1.35).

Remark 2.3 *The De Rham theorem is a difficult result, sometimes used in the mathematical analysis of the Stokes equations (see* e.g. *R. Temam [10]). It states that*

> *If $\boldsymbol{f} \in \big(\mathcal{D}'(\Omega)\big)^d$ $\big(\mathcal{D}'(\Omega)$ denoting the distributions space$\big)$ and $\langle \boldsymbol{f}, \phi \rangle = 0$ for all $\phi \in \big(\mathcal{C}_0^\infty(\Omega)\big)^d$ such that $\operatorname{div}\phi = 0$, then there exists $p \in \mathcal{D}'(\Omega)$ such $\boldsymbol{f} = \boldsymbol{\nabla} p$.*

Theorem 2.2 gives directly a simplified version of the De Rahm theorem that is sufficient for the purpose of this course. Indeed, since $Im(\boldsymbol{\nabla})$ is closed, the closed range theorem yields $Im(\boldsymbol{\nabla}) = \big(Ker(div)\big)^\circ$, which means that

> *If $\boldsymbol{f} \in H^{-1}(\Omega)^d$ and $\langle \boldsymbol{f}, \boldsymbol{v} \rangle = 0$ for all $\boldsymbol{v} \in H_0^1(\Omega)^d$ such that $\operatorname{div}\boldsymbol{v} = 0$, then there exists $p \in L_0^2(\Omega)$ such that $\boldsymbol{f} = \boldsymbol{\nabla} p$.*

2.3 The inf-sup conditions

Here, the abstract framework of Section 2.1 is considered again and practical mathematical tools for verifying points (i) and (ii) in Theorem 2.1 are given. Both of these points are equivalent to an inf-sup condition. However, even if this is somehow artificial (see Remark 2.7 p. 28), they are addressed separately in the next two sections for the sake of clarity.

2.3.1 The inf-sup condition for isomorphisms

A sufficient condition for proving point (i) of Theorem 2.1 is the coercivity on $\operatorname{Ker} B$ which can be stated as

$$\exists \alpha > 0, a(v, v) \geq \alpha \|v\|_X^2, \quad \forall v \in \operatorname{Ker} B.$$

Indeed, in such a case, the Lax-Milgram theorem directly shows that $\Pi \circ A$ is an isomorphism from $V = \operatorname{Ker} B$ onto $V' = (\operatorname{Ker} B)'$. In some problems, the coercivity on V simply results from the coercivity on the whole space X (for example, this is the case for the Stokes equations). But sometimes — for example, in the case of Darcy's equations — $a(\cdot, \cdot)$ is coercive only on V.

The above sufficient condition is sufficient for the purpose of these lecture notes. Nevertheless, the following powerful theorem — which can be skipped during a first reading — is mentioned for the sake of completeness. Applying this theorem with $W = V = \operatorname{Ker} B$ gives a *necessary and sufficient condition* for assessing point (i) of Theorem 2.1.

Theorem 2.3 (Nečas)
Let W be a Banach space and V be a reflexive Banach space. Let $a(\cdot,\cdot)$ be a bilinear continuous form on $W \times V$. Let $f \in V'$. The problem of finding $u \in W$ such that $\forall v \in V$, $a(u,v) = \langle f, v\rangle$ is well-posed if and only if

$$\exists \alpha > 0, \inf_{w \in W} \sup_{v \in V} \frac{a(w,v)}{\|w\|_W \|v\|_V} \geq \alpha, \tag{2.7}$$

and

$$\forall v \in V, \text{ if } a(w,v) = 0, \forall w \in W \text{ then } v = 0. \tag{2.8}$$

In addition, $\forall f \in V'$,

$$\|u\|_W \leq \frac{1}{\alpha} \|f\|_{V'}.$$

Remark 2.4 *Condition (2.8) is equivalent to*

$$\sup_{w \in W} |a(w,v)| > 0, \forall v \neq 0 \in V.$$

Remark 2.5 (Open mapping theorem) *Let us remind a useful consequence of the open mapping theorem. Let X and Y be two Banach spaces, if $A : X \to Y$ is continuous, linear and bijective then A^{-1} is continuous as well. Thus there exists a constant $\alpha > 0$ such that for all $u \in X$, $\alpha \|u\|_X \leq \|Au\|_Y$.*

2.3.2 The inf-sup condition for surjections

Here, an equivalent formulation of point (ii) of Theorem 2.1 is given. This formulation is very important for the applications considered in this course.

Definition 2.1 The property

$$\exists \beta > 0, \inf_{q \in M} \sup_{v \in X} \frac{b(v,q)}{\|v\|_X \|q\|_M} \geq \beta, \tag{2.9}$$

is known as the *inf-sup condition* or the *Babuška-Brezzi condition* or the *LBB condition* (for Ladyzhenskaya-Babuška-Brezzi)[1].

It is convenient to re-write the inf-sup condition (2.9) as

$$\exists \beta > 0, \forall q \in M, \sup_{v \in X} \frac{b(v,q)}{\|v\|_X} \geq \beta \|q\|_M, \tag{2.10}$$

or as

$$\exists \beta > 0, \forall q \in M, \|B^T q\|_{X'} \geq \beta \|q\|_M. \tag{2.11}$$

[1]It is noted that one should rigorously write $\inf_{q \in M \setminus \{0\}}$ and $\sup_{v \in X \setminus \{0\}}$. However, for the sake of simplifying these expressions, the value 0 is excluded each time it appears in a denominator.

The following theorem shows that the inf-sup condition (2.9) is a necessary and sufficient condition for ensuring property (ii) of Theorem 2.1. In particular, it will be shown that this is the practical criterion for verifying that a problem is well-posed after discretization.

Definition 2.2 The orthogonal space to V is defined here as

$$V^{\perp} = \{v \in X, (v, u)_X = 0, \forall\, u \in V\}.$$

Remark 2.6 (Spaces V° and $V^{\perp}$) *The spaces V° and $V^{\perp}$ can be identified through a norm preserving isomorphism (the Riesz representation operator).*

Theorem 2.4
For $\beta > 0$, the three following statements are equivalent

(i) The inf-sup condition (2.9) is satisfied with a constant β.

(ii) B^T is an isomorphism from M onto V° and

$$\forall\, q \in M, \|B^T q\|_{X'} \geq \beta \|q\|_M.$$

(iii) B is an isomorphism from $V^{\perp}$ onto M' and

$$\forall\, u \in V^{\perp}, \|Bu\|_{M'} \geq \beta \|u\|_X.$$

Proof. $(ii) \Rightarrow (i)$: the proof is trivial (see (2.11)).

$(i) \Rightarrow (ii)$: the form (2.11) of the inf-sup condition gives, for all $q \in M$,

$$\|B^T q\|_{X'} \geq \beta \|q\|_M,$$

which shows that B^T is injective and therefore bijective from M on $\operatorname{Im} B^T$. Moreover, it also yields that $(B^T)^{-1}$ is continuous. Indeed, let $f \in \operatorname{Im} B^T$, there exists $q \in M$ such that $f = B^T q$ and

$$\|(B^T)^{-1} f\|_M \leq \frac{1}{\beta} \|f\|_{X'}.$$

Thus $\operatorname{Im} B^T$ is closed (as the inverse range of M by the continuous mapping $(B^T)^{-1}$). Also from the closed range theorem, $\operatorname{Im} B^T = (\operatorname{Ker} B)^{\circ} = V^{\circ}$, which proves that B^T is bijective from M onto V°.

$(ii) \Rightarrow (iii)$ $\operatorname{Im} B^T = V^{\circ}$ is closed thus $\operatorname{Im} B = (\operatorname{Ker} B^T)^{\circ} = M'$. Moreover $V = \operatorname{Ker} B$, thus B restricted to $V^{\perp}$ is injective. Therefore B is an isomorphism from $V^{\perp}$ onto M'. The other implication can be proved with analogous arguments.

As a consequence of the open mapping theorem, $\exists \beta > 0$ such that $\|B^{-1}\| \leq 1/\beta$. But it remains to prove that this β is indeed the same as the β of point (ii).

This results from the isometry between V° and $(V^\perp)'$ and more generally from the isometry between a Hilbert space and its bi-dual.□

The following corollary results from Theorem 2.4.

Corollary 2.3
The following assertions are equivalent

(i) *The inf-sup condition (2.9) is satisfied.*

(ii) $B^T : M \to X'$ *is injective and* B^T *has a closed range.*

(iii) $B : X \to M'$ *is surjective.*

Corollary 2.1, p.24 and Corollary 2.3 yield the following inf-sup condition for the operator $B = -\operatorname{div}$, which is useful for the analysis of the Stokes problem.

Corollary 2.4
There exists $\beta > 0$ *such that*

$$\inf_{q \in L_0^2(\Omega)} \sup_{\boldsymbol{v} \in H_0^1(\Omega)^d} \frac{\displaystyle\int_\Omega q \operatorname{div} \boldsymbol{v}}{\|\boldsymbol{v}\|_1 \|q\|_0} \geq \beta. \tag{2.12}$$

Remark 2.7 *As hinted earlier, the distinction between sections 2.3.1 and 2.3.2 is artificial. There is no "inf-sup condition for isomorphism" and "inf-sup condition for surjection", but a unique inf-sup condition given by Theorem 2.3. Indeed, a surjection becomes a bijection as soon as it is restricted to the orthogonal of its kernel. Thus, if Theorem 2.3 is applied with* $b(\cdot,\cdot)$, $W = (Ker B)^\perp$ *and* $V = M$, *then (2.7) becomes nothing but inequality (iii) in Theorem 2.4, which is equivalent to (2.9) according to Theorem 2.4. The only reason for the chosen presentation style is the separate treatment of the two points of Theorem 2.1.*

2.3.3 Application to mixed problems

From Theorems 2.1 and 2.3 and Corollary 2.3, the following result can be proven.

Theorem 2.5
Problem (2.3) admits a unique solution if and only if

(i) $\Pi \circ A$ *is an isomorphism from* $V = Ker\, B$ *on* V'.

(ii) *The inf-sup condition (2.9) is satisfied.*

When the above conditions are fulfilled, the best constant in the inf-sup condition is denoted by β and the following positive constant[2] *is introduced*

$$\alpha = \inf_{v \in V} \frac{\|\Pi A v\|}{\|v\|}.$$

Then, the unique solution (u, p) of Problem (2.3) satisfies

$$\|u\|_X \leq \frac{1}{\beta}\left(1 + \frac{\|a\|}{\alpha}\right)\|g\|_{M'} + \frac{1}{\alpha}\|f\|_{X'}, \tag{2.13}$$

$$\|p\|_X \leq \frac{\|a\|}{\beta^2}\left(1 + \frac{\|a\|}{\alpha}\right)\|g\|_{M'} + \frac{1}{\beta}\left(1 + \frac{\|a\|}{\alpha}\right)\|f\|_{X'}. \tag{2.14}$$

2.4 Minimization with constraints

It was shown in Theorem 1.2, p. 9 that when the bilinear form $a(\cdot, \cdot)$ is *symmetric* $\big(a(u, v) = a(v, u),\ \forall u, v \in X\big)$ and *positive* $\big(a(v, v) \geq 0, \forall v \in X\big)$, the variational problem (1.13) can be interpreted as the minimization of an energy functional on the *whole space* X. Here, it is shown that mixed problems such as (2.3) are related to the minimization of the same energy functional subject to the *constraint* that the solution belongs to the set $V(g)$ defined by

$$V(g) = \{u \in X, Bu = g\} = \{u \in X, b(u, q) = \langle g, q\rangle, \forall\, q \in M\}.$$

Note that the space V defined in (2.5), p. 22 coincides with $V(0)$.

Definition 2.3 The application $J : X \to \mathbb{R}$ defined by

$$J(v) = \frac{1}{2}a(v, v) - \langle f, v\rangle$$

is called the *energy functional* of problem (2.3).

Definition 2.4 The application $\mathcal{L} : X \times M \to \mathbb{R}$ defined by

$$\mathcal{L}(v, q) = J(v) + b(v, q) - \langle g, q\rangle$$

is called the *Lagrangian* of problem (2.3).

Definition 2.5 A pair $(u, p) \in X \times M$ is said to be a *saddle point* of the Lagrangian $\mathcal{L}$ if

$$\mathcal{L}(u, q) \leq \mathcal{L}(u, p) \leq \mathcal{L}(v, p), \quad \forall (v, q) \in X \times M.$$

[2] See Remark 2.5, p.26.

Proposition 2.1
If (u,p) is a saddle point of $\mathcal{L}$ then

$$\mathcal{L}(u,p) = \min_{v\in X}\max_{q\in M}\mathcal{L}(v,q) = \max_{q\in M}\min_{v\in X}\mathcal{L}(v,q). \tag{2.15}$$

Proof. First, it is noted that

$$\sup_{q\in M}\inf_{v\in X}\mathcal{L}(v,q) \le \inf_{v\in X}\sup_{q\in M}\mathcal{L}(v,q). \tag{2.16}$$

Indeed,

$$\inf_{v'\in X}\mathcal{L}(v',q) \le \mathcal{L}(v,q) \le \sup_{q'\in M}\mathcal{L}(v,q').$$

Therefore,

$$\sup_{q'\in M}\inf_{v'\in X}\mathcal{L}(v',q) \le \sup_{q'\in M}\mathcal{L}(v,q').$$

Taking the inf over X of the above expression leads to (2.16).

Next, assume that (u,p) is a saddle point of $\mathcal{L}$. Taking the sup over M (resp. inf over X) in the lower bound (resp. upper bound) of the definition of a saddle point yields

$$\sup_{q\in M}\mathcal{L}(u,q) \le \mathcal{L}(u,p) \le \inf_{v\in X}\mathcal{L}(v,p).$$

Thus,

$$\inf_{v\in X}\sup_{q\in M}\mathcal{L}(v,q) \le \sup_{q\in M}\inf_{v\in X}\mathcal{L}(v,q).$$

Note that this relation results from the fact that $\mathcal{L}$ admits a saddle point. Since (2.16) is also true, it follows that

$$\inf_{v\in X}\sup_{q\in M}\mathcal{L}(v,q) = \sup_{q\in M}\mathcal{L}(u,q) = \mathcal{L}(u,p) = \inf_{v\in X}\mathcal{L}(v,p) = \sup_{q\in M}\inf_{v\in X}\mathcal{L}(v,q).$$

□

Theorem 2.6
If $a(\cdot,\cdot)$ is symmetric and positive, then the following two assertions are equivalent

(i) *(u,p) is a saddle point of $\mathcal{L}$.*

(ii) *(u,p) is a solution of (2.3).*

Proof. First, note that

$$\begin{array}{rrcl} & \mathcal{L}(u,q) & \le & \mathcal{L}(u,p), \quad \forall q\in M \\ \Leftrightarrow & b(u,q-p) & \le & \langle g, q-p\rangle, \quad \forall q\in M \\ \Leftrightarrow & b(u,tr) & \le & \langle g, tr\rangle, \quad \forall r\in M, \forall t\in\mathbb{R} \\ \Leftrightarrow & b(u,r) & = & \langle g, r\rangle, \quad \forall r\in M \end{array}$$

and hence the second equation of problem (2.3).

Next, for a given $p \in M$, define $\tilde{J}$ and $\tilde{f}$ by $\tilde{J}(v) = \frac{1}{2}a(v,v) - \langle \tilde{f}, v\rangle$, $\langle \tilde{f}, v\rangle = \langle f, v\rangle - b(v,p)$, $\forall v \in X$. Then,

$$\begin{array}{rrcl} & \mathcal{L}(u,p) & \leq & \mathcal{L}(v,p), \quad \forall v \in X \\ \Leftrightarrow & \tilde{J}(u) & \leq & \tilde{J}(v) \quad \forall v \in X \\ \Leftrightarrow & a(u,v) & \leq & \langle \tilde{f}, v\rangle, \quad \forall v \in X \\ \Leftrightarrow & a(u,v) + b(v,p) & = & \langle f, v\rangle, \quad \forall v \in X \end{array}$$

and hence the first equation of problem (2.3). $\diamondsuit$

Theorem 2.7
Assume that the bilinear form $a(\cdot,\cdot)$ is symmetric and coercive and that $V(g)$ is non-empty. Then, the problem of finding $u \in X$ such that

$$J(u) = \inf_{v \in V(g)} J(v) \tag{2.17}$$

has a unique solution.

Furthermore, if the bilinear form $b(\cdot,\cdot)$ satisfies the inf-sup condition (2.9), there exists a unique $p \in M$ such that (u,p) is the unique saddle point of $\mathcal{L}$ and the unique solution of the mixed problem (2.3).

Proof. Let $u_g \in V(g)$. From the linearity of B, $V(g) = u_g + V$. Thus the minimization problem (2.17) is equivalent to the minimization of $\widetilde{J}(v) = J(u_g + v)$ over the *linear space* V. The space V is closed, as it is the inverse range of the closed space $\{0\}$ by the continuous mapping B. Hence, it is a Hilbert space for the norm $\|\cdot\|_X$. According to the Lax-Milgram theorem and Theorem 1.2, p. 9 applied to the energy $\widetilde{J}$ in $\left(V, \|\cdot\|_X\right)$, the minimization problem (2.17) has a unique solution $u = u_g + u_0$, where $u_0 \in V$ is the unique solution of

$$a(u_0, v) = \langle f, v\rangle - a(u_g, v), \quad \forall v \in V,$$

which proves the first part of the theorem.

The inf-sup condition (2.9) implies that B is surjective. Thus, B has in particular a closed range, and therefore $V^\circ \overset{def}{=} (\operatorname{Ker} B)^\circ = \operatorname{Im} B^T$. Let $u \in V(g)$ be the unique solution of the minimization problem (2.17). Since $a(u,v) = \langle f, v\rangle$ for all $v \in V$, $f - Au \in V^\circ$ and thus $f - Au \in \operatorname{Im} B^T$. Hence, there exists $p \in M$ such that $Au + B^T p = f$, and this p is unique by injectivity of B^T. Since also $Bu = g$, the pair (u,p) is the unique solution of the mixed problem (2.3) and therefore, according to theorem (2.6), is the unique saddle point of $\mathcal{L}$.$\square$

Remark 2.8 *The reader is reminded that the inf-sup condition (2.9) is equivalent to the surjectivity of B, which is itself equivalent to saying that B has a closed range and B^T is injective. If in Theorem 2.7 one assumes only that B has a closed* *range, then one can only prove the existence of the Lagrange multiplier p but not its uniqueness.*

Remark 2.9 *The* dual energy *is defined by*

$$J^*(q) = \inf_{v \in X} \mathcal{L}(v, q) = \mathcal{L}(v(q), q).$$

where $v(q)$ is the solution of the minimization of $\mathcal{L}(v, q)$ over X. If (u, p) is a saddle point of $\mathcal{L}$, then

$$\mathcal{L}(u, p) = \sup_{q \in M} \inf_{v \in X} \mathcal{L}(v, q) = \sup_{q \in M} \mathcal{L}(v(q), q) = \sup_{q \in M} J^*(q).$$

The Uzawa algorithms *are gradient methods applied to the maximization of the dual energy.*

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